1. a) Define the sequence $\{v_n\}$ such that $v_n \longrightarrow v_0$

Define $u_n = A^{-1} v_n$, $u = A^{-1} v \Rightarrow v_n = A u_n$, $v = Au$.

$$
\begin{aligned}
a(\|u_n-u\|)\|A^{-1}v_n - A^{-1}v\| &= a(\|u_n-u\|)\|u_n - u\| \\
&\le \langle Au_n - Au, u_n - u\rangle \\
&= \langle v_n - v, u_n - u\rangle \\
&\le \underbrace{\|v_n - v\|}_{\to 0}\|u_n - u\|
\end{aligned}
$$

$$\Rightarrow a(\|u_n-u\|)\|A^{-1}v_n - A^{-1}v\| \to 0$$

b) As $\|u_n - u\| \to 0 \Rightarrow \exists n_0$ s.t $\|u_n - u\| \le t_0 \ \forall n \ge n_0$

Following Theorem 2.31, part 3, proof

we have $\|u_n - u\| \to 0$ for $n \to \infty$.

For any $v_n \in X$

$$0 = \langle Au_n - Au, u_n - v_n\rangle = \langle Au_n - Au, u_n - u\rangle + \langle Au_n - Au, u - v_n\rangle$$

Then,

$$
\begin{aligned}
\|u_n - u\|^{1+\alpha} &\le a(\|u_n-u\|)\|u_n - u\| \\
&\le \langle Au_n - Au, u_n - u\rangle \\
&\le -\langle Au_n - Au, u - v_n\rangle \\
&\le \|u - v_n\| \, L \|u_n - u\| \qquad \forall v \in X_n
\end{aligned}
$$

$$\Rightarrow \|u_n - u\|^{1+\alpha} \le L \liminf_{v \in X_n} \|u_n - u\|$$

$$\Rightarrow \|u_n - u\| \le \left[L \liminf_{v \in X_n} \|u_n - u\|\right]^{1/\alpha} \quad \text{for } n \ge n_0.$$

2. If we look for a fixed point u of $u = T_t u$, $T_t x = x - t\mathcal{U}^{-1}(Ax-b) \quad \forall x \in X$

$$\Leftrightarrow \quad u = u - t\mathcal{U}^{-1}(Au-b)$$
$$\Leftrightarrow \quad 0 = -t\mathcal{U}^{-1}(Au-b)$$
$$\Leftrightarrow \quad \mathcal{U}^{-1}(Au-b) = 0$$
$$\Leftrightarrow \quad Au-b = 0$$
$$\Leftrightarrow \quad Au = b$$

So fixed point of $u = T_t u$ equivalent to $Au=b$

Consider

$$\|T_t x - T_t y\|^2 = \|x - t\mathcal{U}^{-1}(Ax-b) - y + t\mathcal{U}^{-1}(Ay-b)\|$$
$$= \|x-y - t\left(\mathcal{U}^{-1}(Ax-b) - \mathcal{U}^{-1}(Ay-b)\right)\|$$
$$= \|x-y\|^2 - 2t\left(x-y, \mathcal{U}^{-1}(Ax-b) - \mathcal{U}^{-1}(Ay-b)\right) + t^2\|\mathcal{U}^{-1}(Ax-b) - \mathcal{U}^{-1}(Ay-b)\|$$
$$= \|x-y\|^2 - 2t\left(\mathcal{U}(x-y), Ax-Ay\right) + t^2\|Ax-Ay\|^2$$
$$\leq (1 - 2tM + t^2 L^2)\|x-y\|$$

$$\Rightarrow \|T_t x - T_t y\| \leq k(t)\|x-y\|^2$$

where $k(t) = (1 - 2Mt + L^2 t^2)^{1/2} < 1$.

As T_t is contractive then by Banach's fixed point theorem a unique fixed point $u = T_t u$ exists $\Rightarrow$ unique solution of $Au=b$; furthermore the sequence

$$v_{i+1} = v_i - t\mathcal{U}^{-1}(Ax-b) \Leftrightarrow \mathcal{U}(v_{i+1}) = \mathcal{U}(v_i) - t\mathcal{U}^{-1}(Ax-b)$$

converges to u with error

$$\|u - v_i\| \leq \frac{k^i}{1-k}\|v_1 - v_0\| = \frac{k^i}{1-k}\|v_0 - t\mathcal{U}^{-1}(Ax-b) - v_0\|$$
$$= \frac{k^i t}{1-k}\|Ax-b\|$$

$\square$

3. $(k(t))^2 = (1 - 2Mt + L^2t^2)$

$$0 = \frac{d}{dt} k(t)^2 = \frac{d}{dt}(1-2Mt+L^2t^2) = -2M + 2L^2t$$

Therefore at $t_0 = \frac{M}{L^2}$. $k(t)$ achieves its minimum,

and $k_0 = k(t_0) = \left(1 - \frac{2M^2}{L^2} + \frac{M^2L^2}{L^4}\right)^{1/2}$

$$= \left(1 - \frac{2M^2}{L^2} + \frac{M^2}{L^2}\right)^{1/2}$$

$$= \left(1 - \frac{M^2}{L^2}\right)^{1/2}$$

4. For all $n \in \mathbb{N}$, $A_n \in X_n$. As P_n linear operator $P_nu = u$

$$\|A_nu - A_nv\| = \|P_n^* A P_n u - P_n^* A P_n v\|$$

$$= \langle AP_nu - AP_nv, AP_nu - AP_nv\rangle$$

$$= \langle Au - Av, Au - Av\rangle$$

$$= \|Au - Av\|$$

$$\leq L\|u - v\|$$

$\Rightarrow A_n$ is Lipschitz continuous

Similarly $\langle A_nu - A_nv, u - v\rangle = \langle AP_nu - AP_nv, P_nu - P_nv\rangle$

$$= \langle Au - Av, u - v\rangle$$

$$\geq M\|u - v\|^2$$

$\Rightarrow A_n$ is strongly monotone

Hence, Theorem 2.32 can be applied to show Theorem 2.33. □

5. As in question 2 can show that U_n is strongly contractive for all n. Then, by Lemma 2.35 the sequence $w_n = U_n w_{n-1} \Leftrightarrow \mathcal{U}^{-1}(w_n) = \mathcal{U}^{-1}(w_{n-1}) - t(A_nw - b_n)$ converges to u.