Nonlinear Functional Analysis

Practicals

30th April 2020

Recap

Theorem 2.32. *Let X be a real separable Hilbert space with inner product $(\cdot,\cdot)$. Suppose that the operator $A : X \to X^*$ satisfies the conditions:*

1. *The operator A is strongly monotone; i.e, there exists a constant $M > 0$ such that for any $x, y \in X$*
$$\langle Ax - Ay, x - y\rangle \geq M\|x - y\|^2.$$
2. *The operator A is Lipschitz continuous; i.e, there exists a constant $L > 0$ such that for any $x, y \in X$*
$$\|Ax - Ay\| \leq L\|x - y\|.$$

Choose $b \in X^*$; then, for any $v_0 \in X$, $t \in (0, 2ML^{-2})$, we can construct the sequence $\{v_i\}$ defined by

$$\mathcal{U}(v_i) = \mathcal{U}(v_{i-1}) - t(Av_{i-1} - b), \qquad i = 1, 2, \ldots.$$

The sequence $\{v_i\}$ converges to the unique solution u of $Au = b$ with error

$$\|v_i - u\| \leq \frac{k^i t}{1-k}\|Av_0 - b\|,$$

where

$$k(t) = (1 - 2Mt + L^2t^2)^{1/2} < 1.$$

Theorem 2.33. *Assuming the conditions of Theorem 2.32, for every $n \in \mathbb{N}$ there exists a unique solution u_n to the equation $A_n u_n = b_n$ in the space X_n. For any $v_{n,0} \in X$, $t \in (0, 2ML^{-2})$ we construct the sequence $\{v_{n,i}\}$ by*

$$\mathcal{U}_n(v_{n,i}) = \mathcal{U}_n(v_{n,i-1}) - t(A_n v_{n,i-1} - b_n), \qquad i = 1, 2, \ldots,$$

where $\mathcal{U}_n : X_n \to X_n^*$ is the dualisation for the space X_n. Then, the sequence $\{v_{n,i}\}$ converges for $i \to \infty$ to a unique solution u_n of the equation $A_n u_n = b_n$ with error

$$\|v_{n,i} - u_n\| \leq \frac{k^i t}{1-k}\|A_n v_{n,0} - b_n\| \leq \frac{k^i t}{1-k}\|Av_{n,0} - b\|,$$

where

$$k(t) = (1 - 2Mt + L^2t^2)^{1/2} < 1.$$

Lemma 2.34. *Let Z be a Banach space space with norm $\|\cdot\|$, $\{Z_n\}$ an increasing sequence of closed subspaces*

$$Z_1 \subset Z_2 \subset \cdots \subset Z_n \subset Z,$$

and there exists a sequence of operators $\{U_n\}$, $U_n : Z_n \to Z_n$ which are strongly contractive in the sense that there exists a constant $k \in (0, 1)$ such that

$$\|U_n u - U_n v\| \leq k\|u - v\|, \qquad \forall u, v \in Z_n, n = 1, 2, \ldots.$$

Suppose that the sequence of fixed points $\{u_n\}$, $U_n u_n = u_n$ converges in the norm of the space Z to $u \in Z$. Then, for any $v_0 \in Z$ the sequence $\{v_i\}$ defined by

$$v_i = U_i v_{i-1}, \qquad i = 1, 2, \ldots$$

converges to u.

Theorem 2.35. *Assuming the conditions of Theorem 2.32, and with the notation from Theorem 2.33. Then, the* projection-iteration *sequence w_n constructed according to*

$$\mathcal{U}_n(w_n) = \mathcal{U}_n(w_{n-1}) - t(A_n w_{n-1} - b_n), \qquad n = 1, 2, \ldots,$$

where $\mathcal{U}_n : X_n \to X_n^*$ is the dualisation for the space X_n, converges for any $w_{n,0} \in X$ and $t \in (0, 2ML^{-2})$ to a unique solution u of the equation $Au = b$.

Exercises

1. Let X be a reflexive separable Banach space and the sequence of finite dimensional subspaces $X_n \subset X$, $n \in \mathbb{N}$ is dense in the space X in the limit.
   (a) If the operator $A : X \to X$ is uniformly monotone; i.e., there exists an increasing $a : \mathbb{R}_+ \to \mathbb{R}_+$, $a(0) = 0$, such that for all $u, v \in X$

   $$\langle Au - Av, u - v \rangle \geq a(\|u - v\|)\|u - v\|,$$

   then prove that the operator A^{-1} is continuous.
   (b) Suppose that for the function $a(t)$, $t \in \mathbb{R}_+$, in the definition of uniform monontonicity,

   $$a(t)t \geq t^{1+\alpha} \qquad \text{for } 0 < t \leq t_0, t_0 > 0, \alpha > 0,$$

   and that the operator A is Lipschitz continuous; then, prove that there exists an n_0 such that the error of the Galerkin approximation u_n (in the space X_n) to the exact solution of the problem $Au = b$ is given by the estimate

   $$\|u_n - u\| \leq \left(L \inf_{v \in X_n} \|v - u_n\| \right)^{\frac{1}{\alpha}} \qquad \text{for } n \geq n_0.$$

2. Prove Theorem 2.32

   Hint. Show that finding solution to $Au = b$ is equivalent to finding a fixed point of T_t:

   $$Au = b \qquad \Longleftrightarrow \qquad u = T_t u, \quad T_t x = x - t\mathcal{U}^{-1}(Ax - b)\, \forall x \in X.$$

   Then, show that Banach's fixed point theorem can be applied to T_t.

3. Let the assumptions of Theorem 2.32 be met. Show that the function $k(t)$ acquires its minimum in the interval $(0, 2ML^{-2})$ at $t = t_0$, where

   $$t_0 = \frac{M}{L^2}, \qquad \text{and} \qquad k_0 = k(t_0) = \left(1 - \left(\frac{M}{L} \right)^2 \right)^{\frac{1}{2}}.$$

4. Prove Theorem 2.33.

   Hint. Verify that A_n, for all $n \in \mathbb{N}$, is strongly monotone and Lipschitz continuous. Then apply Theorem 2.32 for $A_n : X_n \to X_n^*$.

5. Prove Theorem 2.35.

   Hint. The projection-iteration sequence w_n can be written as $w_n = U_n w_{n-1}$, where

   $$U_n w = w - t\mathcal{U}^{-1}(A_n w - b_n) \qquad \text{for all } w \in X_n.$$

   Then apply Lemma 2.34.