Corollary 2.28 - Proof

Lemma 2.4/2.6/2.7/2.10

1) Assume A meets (1); i.e., A is coercive, demicontinuous, bounded, & satisfies (S). In order to ~~meet~~ Theorem 2.27 we need to show A is continuous on finite dimensional subspaces & satisfies $(M)_0$.

- From Lemma 2.6 demicontinuous $\Rightarrow$ cont. on finite dim. subspaces
- From Lemma 2.10 (S) holds $\Rightarrow$ $(S)_0$ holds;

Then, by Lemma 2.10 $(S)_0$ holds & demicontinuous $\Rightarrow$ $(M)_0$ holds

$\Rightarrow$ Conditions of Theorem 2.27 met.

2) Now assume A meets (2); i.e., A is coercive, pseudomonotone & bounded. To meet Theorem 2.27 need to show A is continuous on finite dimensional subspaces & satisfies $(M)_0$.

From Lemma 2.10: pseudomonotone $\Rightarrow$ (M) holds $\Rightarrow$ $(M)_0$ holds

and also: pseudomonotone & bounded $\Rightarrow$ demicontinuous

Then, from Lemma 2.6: demicontinuous $\Rightarrow$ cont. on finite dim sspaces

$\Rightarrow$ Conditions of Theorem 2.27 met $\square$

Theorem 1 - Proof

- First prove $B: X \to X^*$ monotone & radially continuous satisfies $(M)_0$. As B is monotone, from Lemma 2.7 (4)

B radially continuous $\Leftrightarrow$ B hemicontinuous

From Lemma 2.10

B hemicontinuous & monotone $\Rightarrow$ B pseudomonotone

B pseudomonotone $\Rightarrow$ (M) satisfied

(M) satisfied $\Rightarrow$ $(M)_0$ satisfied.

- Now show A satisfies $(M)_0$. Assume $u_n \rightharpoonup u$, $Au_n \rightharpoonup b$,

$$\limsup_{n\to\infty} \langle Au_n, u_n\rangle = \langle b, u\rangle$$

$$\forall v \in X \quad \langle Bu_n, v\rangle = \underbrace{\langle Au_n, v\rangle}_{\to \langle b, v\rangle} - \underbrace{\langle Tu_n, v\rangle}_{\to \langle Tu, v\rangle} \to \langle b - Tu, v\rangle$$

as T weakly continuous.

$$\Rightarrow \quad Bu_n \rightharpoonup b - Tu$$

$$\limsup_{n\to\infty} \langle Bu_n, u_n\rangle = \underbrace{\limsup_{n\to\infty} \langle Au_n, u_n\rangle}_{=\langle b,u\rangle} - \limsup_{n\to\infty} \langle Tu_n, u_n\rangle$$

$$\limsup \langle Tu_n, u_n\rangle \geq \liminf \langle Tu_n, u_n\rangle \geq \langle Tu, u\rangle \quad \left[\begin{array}{l}T \text{ weakly} \\ \text{lower semi-cont.}\end{array}\right]$$

$$\Rightarrow \quad -\limsup \langle Tu_n, u_n\rangle \leq -\langle Tu, u\rangle$$

$$\Rightarrow \limsup \langle Bu_n, u_n\rangle \leq \langle b - Tu, u\rangle$$

Therefore, as left hand side of (M) holds for $B \Rightarrow$

$$Bu = b - Tu \Rightarrow Bu + Tu = b \Rightarrow Au = b;$$

hence, A satisfies $(M)_0$

- Also need to show A is cont. on finite dimensional subspaces

From Lemma 2.6 T weakly continuous $\Rightarrow T$ demicontinuous

(This may be missing in the lecture notes but its true as strong convergence $\Rightarrow$ weak convergence).

& From Lemma 2.7 as B monotone & radially continuous

$\Rightarrow$ B demicontinuous

Then can show A also demicontinuous: Assume $u_n \to u$

$$\forall v \quad \langle Au_n, v\rangle = \underbrace{\langle Bu_n, v\rangle}_{\to \langle Bu, v\rangle \text{ by demicontinuity}} + \underbrace{\langle Tu_n, v\rangle}_{\to \langle Tu, v\rangle} \to \langle Bu - Tu, v\rangle = \langle Au, v\rangle$$

$$\Rightarrow Au_n \rightharpoonup Au.$$

As A demicontinuous $\Rightarrow$ cont. on finite dim. subspaces (Lemma 2.6).

- Conditions of Theorem 2.27 are met $\Rightarrow Au=b$ has a solution.

- Consider the sequence $\{u_n\}$ s.t $u_n \rightharpoonup u$; where $Au_n = b$.

  As $Au_n = b \Rightarrow Au_n \rightharpoonup b$

  $\&$ $\limsup_{n\to\infty} \langle Au_n, u_n\rangle = \limsup_{n\to\infty} \langle b, u_n\rangle = \langle b, u\rangle$

  So $u_n \rightharpoonup u$, $Au_n \rightharpoonup b$, and $\limsup_{n\to\infty} \langle Au_n, u_n\rangle = \langle b, u\rangle$;

  therefore, as A satisfies (M)

  $$Au=b$$

  $\Rightarrow$ ~~For~~ a sequence $\{u_n\} \in K$, $K = \{u \in X, Au=f\} \subset X$,

  $$u_n \rightharpoonup u \Rightarrow u \in K$$

  $\Rightarrow K$ (set of all solutions of $Au=b$) is weakly closed.