Nonlinear Functional Analysis

Practicals

23th April 2020

Prove the following.

Corollary 2.28. *Let X be a separable, reflexive Banach space and the operator $A : X \to X^*$ meets one of the following conditions:*

1. *A is coercive, demicontinuous, bounded, and satisfies condition (S),*
2. *A is coercive, pseudomonotone, and bounded.*

Then, the operator A is surjective. In other words, a solution of the equation $Au = b$ exists for every right-hand side $b \in X^*$. Additionally, A^{-1} is bounded; i.e., there exists a function $N : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $u \in X$,

$$\|u\| \leq N(\|Au\|).$$

Hint. Look at Lemmas 2.4, 2.6, 2.7, and 2.10.

Theorem 1. *Let X be a reflexive separable Banach space, the operator $A : X \to X^*$ is coercive, and the decomposition*

$$A = B + T$$

exists, where the operator $B : X \to X^*$ is monotone, radially continuous, and the operator $T : X \to X^*$ is weakly continuous. If the function $\varphi(x) := \langle Tx, x\rangle$ is weakly lower semicontinuous (see Definition 1.8 with a weak topology in the space X); then, the equation $Au = b$ has a solution for every right-hand side $b \in X^*$. Additionally, the set of all solutions for a fixed right-hand side is weakly closed.

Hint. First prove that B satisfies (M) and, therefore, $(M)_0$ (use Lemmas 2.7 and 2.10). Then it is possible to show that A also satisfies the condition and Theorem 2.27 can be applied to show existence of the solution. Finally, show the solutions are weakly closed.