1. Select $K = H$

From Stampacchia we have that for any $F \in H^*$ $\exists\, u \in H$ such that

$$a(u, v-u) \geq F(v-u) \quad \text{for all } v \in H$$

Select $v = w + u \ \forall w \in H$

$$a(u, w) \geq F(w) \qquad (1)$$

& $v = -w + u, \ \forall w \in H$

$$a(u, -w) \geq F(-w)$$

$$\Rightarrow \qquad F(w) \geq a(u, w) \qquad (2)$$

Then, by (1) & (2) $\exists\, u \in H$ s.t.

$$a(u, w) = F(w) \quad \text{for all } w \in H.$$

2. As $a(\cdot,\cdot)$ continuous bilinear form, $v \to a(u, v)$ is continuous; hence, by Riesz-Frechét $\exists\, Au$ (depends on u) s.t.

$$a(u, v) = (Au, v) \qquad \forall v \in H$$

Then,

$$\|Au\| \leq c_1 \|u\| \ \ \& \ \ (Au, u) \geq c_2 \|u\|^2$$

Let f be representation of continuous linear functional $F \in H^*$; i.e. $F(v) = (v, f)$ for all $v \in H$.

Construct $P_K : H \to K$ as

$$\|P_K w - w\| = \min_{v \in K} \|w - v\|$$

From convex set theory,

$$(w - P_K w, v - P_K w) \leq 0 \text{ for } v \in K, \ \|P_K w_1 - P_K w_2\| \leq \|w_1 - w_2\|$$

K closed $\Rightarrow (P_K w - w, v) = 0$ for $v \in K$

· Show that $\forall \alpha > 0$ following equivalent:

(1) $\alpha (Au, v-u) \geq \alpha (f, v-u)$ for $v \in K$

(2) $(\alpha f - \alpha Au , v-u) \leq 0$ for $v \in K$

(3) $u = P_K(\alpha f - \alpha Au + u)$

· (1) $\Leftrightarrow$ (2):

$$\alpha (Au, v-u) \geq \alpha (f, v-u)$$

$$\Leftrightarrow \quad 0 \geq (\alpha f, v-u) - (\alpha Au, v-u)$$

$$= (\alpha f - \alpha Au, v-u)$$ by linearity of inner product (first term).

· (2) $\Rightarrow$ (3): Let $w = \alpha f - \alpha Au + u$

$$0 \leq \|P_K w - u\|^2 = (P_K w - u, P_K w - u) \quad [(P_K w - w, v) = 0 \text{ for } v \in K]$$

$$= (w - u, P_K w - u)$$

$$= (\alpha f - \alpha Au, P_K w - u)$$

$$\leq 0$$

$$\Rightarrow \|P_K w - u\|^2 = 0 \Rightarrow u = P_K w$$

· (3) $\Rightarrow$ (2)

P_K has property that

$$(w - P_K w, v - P_K w) \leq 0 \quad \text{for } v \in K$$

Select $w = \alpha f - \alpha Au + u$, $u = P_K(\alpha f - \alpha Au + u) = P_K w$

then

$$(\alpha f - \alpha Au + u - u, v-u) \leq 0$$

$$\Rightarrow (\alpha f - \alpha Au, v-u) \leq 0$$

Define the mapping $S: K \to K$ by $Sv = P_K(\alpha f - \alpha Av + v)$

Consider $x, y \in K$

$$\|Sx - Sy\|^2 = \|P_K(\alpha f - \alpha Ax + x) - P_K(\alpha f - \alpha Ay + y)\|^2$$
$$\leq \|\alpha f - \alpha Ax + x - \alpha f + \alpha Ay - y\|$$
$$= \|\alpha Ax - \alpha Ay\|^2 - (\alpha Ax - \alpha Ay, x - y)$$
$$- (x - y, \alpha Ax - \alpha Ay) + \|x - y\|^2$$
$$= \alpha^2 \|A(x-y)\|^2 - \alpha(A(x-y), x-y)$$
$$- \alpha(x - y, A(x-y)) + \|x-y\|^2$$
$$\leq \alpha^2 c_1^2 \|x-y\|^2 - 2\alpha c_2 \|x-y\|^2 + \|x-y\|^2$$
$$= (\alpha^2 c_1^2 - 2\alpha c_2 + 1)\|x-y\|^2$$
$$\leq k^2 \|x-y\|^2$$

where $k^2 = (\alpha^2 c_1^2 - 2\alpha c_2 + 1)$. If $\alpha c_1^2 < 2c_2$ then

$$k^2 < 2\alpha^2 c_1^2 - 2\alpha c_2 + 1 = 1.$$

Hence, by Banach's fixed point theorem there exists a $u \in K$ such that $u = P_K(\alpha f - \alpha Au + u)$.

Then by above equivalence

$$\Rightarrow \alpha(Au, v-u) \geq \alpha(f, v-u)$$
$$\Rightarrow (Au, v-u) \geq (f, v-u)$$
$$\Rightarrow a(u, v-u) \geq F(v-u).$$

which proves the first statement.

- As $a(u,v)$ is symmetric then it induces a norm in H

$$\|u\|_a^2 \equiv a(u,u)$$

$$c_2\|u\|^2 \leq (Au, u) = a(u,u) = \|u\|_a$$

$$\|u\|_a = a(u,u) = (Au, u) \leq \|Au\|\|u\| \leq c_1\|u\|^2$$

$\Rightarrow$ Norms are equivalent.

From Riesz-Frechet $\exists g$ s.t

$$F(v) = a(g,v) \qquad \forall v \in H$$

$$a(g-u, v-u) = a(g, v-u) - a(u, v-u)$$
$$= F(v-u) - a(u, v-u) \qquad \forall v \in K$$

By first part $\quad a(u, v-u) \geq F(v-u)$

$$\Rightarrow F(v-u) - a(u, v-u) \leq 0$$

$$\Rightarrow a(g-u, v-u) \leq 0 \qquad \forall v \in K$$

By Proposition 1.5, point 8 as K, none-empty, closed & Convex subset of Hilbert space then

$$\|g-u\|_a = \min_{v \in K} \|g-v\|_a$$

$$\Rightarrow a(g-u, g-u) = \min_{v \in K} a(g-v, g-v)$$

$$\Rightarrow \cancel{a(g,g)} - \underbrace{2a(g,u)}_{a \text{ symmetric}} + a(u,u) = \min_{v \in K}\left(\cancel{a(g,g)} - 2a(g,v) + a(v,v)\right)$$

$$a(u,u) - 2F(u) = \min_{v \in K} \left(a(v,v) - 2F(v)\right)$$

Divide through by 2 completes the proof $\square$.

3. Select $X = c_0 = \{x = \{x_i\} : x_i \to 0 \text{ for } i \to \infty\}$,

$$\|x\| = \max_i |x_i|$$

M - closed ball in space $\to$ clearly non-empty, bounded & closed

Define A as $\quad Ax = \{1, x_1, x_2, \ldots\}$, for $x = \{x_1, x_2, \ldots\} \in M$.

$$\|Ax - Ay\| = \max(|1-1|, |x_1 - y_1|, |x_2 - y_2|, \ldots)$$
$$= \max(|x_1 - y_1|, |x_2 - y_2|, \ldots)$$
$$= \|x - y\| \qquad \Rightarrow \text{non-expensive.}$$

Assume a fixed point exists; i.e, $\exists x \in c_0$ s.t.

$$Ax = x$$

$$\Rightarrow \{1, x_1, x_2, \dots\} = \{x_1, x_2, x_3, \dots\}$$

$$\Rightarrow 1 = x_1 = x_2 = x_3 = \dots$$

$$\Rightarrow x_i \not\to 0 \quad \Rightarrow x \notin c_0.$$

And, hence, we have a contradiction. Therefore, a fixed point does not exist.

4. $B(w,r)$ - closed ball in Hilbert space, centre $w \in H$, radius $r > 0$.

$T : B(w,r) \to H$ - monotone & continuous on $B(w,r) \cap M$

M - arbitrary finite dimensional subspace.

Assume for each $z \in S(w,r) = \{u \in H : \|u - w\| = r\}$

$$z - w + \lambda Tz \neq 0 \quad \text{for all } \lambda > 0. \quad (*)$$

Define operator $A : B \to H$:

$$Ax = T(xr + w), \ x \in B$$

By Theorem 2.22 $\exists x_0 \in B$ s.t $(Ax_0, y - x_0) \geq 0$ for all $y \in B$. Moreover if $\|x_0\| < 1$ then $Ax_0 = 0$. If $\|x_0\| = 1$ $Ax_0 = 0$ if for all $x \in \{u \in H : \|u\| = 1\}$

$$x + \lambda Ax \neq 0 \text{ for all } \lambda \geq 0.$$

By condition $(*)$ $\forall z = xr + w \in S(w,r), x \in B$

then $\|z - w\| = \|xr + w - w\| = r\|x\| = r$

$\Rightarrow \|x\| = 1 \quad \Rightarrow x \in \{u \in H : \|u\| = 1\}$

& $z - w + \lambda Tz = xr + w - w + \lambda Ax$

$= xr + \lambda Ax \neq 0 \quad \forall \lambda \geq 0$

$\Rightarrow x + \lambda_r x \neq 0, \ \forall \lambda_r = \frac{\lambda}{r} \geq 0;$

therefore condition of Theorem 2.22 met & $Ax_0 = 0$

$\Rightarrow T(x_0 r + w) = 0 \quad \Rightarrow \exists z_0 = x_0 r + w \in B(w,r)$ s.t $Tz_0 = 0$ $\square$