Nonlinear Functional Analysis

Practicals

9th April 2020

Recap

Theorem 2.16 (Lax-Milgram). *Let H be a real Hilbert space, and $a(u,v)$ a continuous and coercive bilinear form defined on H; i.e, there exists constants $c_1 > 0$ and $c_2 > 0$ such that for all $u, v \in H$*

$$|a(u,v)| \leq c_1 \|u\|\|v\| \qquad \text{and} \qquad a(u,v) \geq c_2 \|u\|^2.$$

Then, for any $F \in H^*$, there exists $u \in H$ such that

$$a(u,v) = F(v) \qquad \text{for all } v \in H. \tag{1}$$

Moreover, if $a(\cdot,\cdot)$ is symmetric, then for any u satisfying (1) above

$$\frac{1}{2}a(u,u) - F(u) = \min_{v \in H} \left\{ \frac{1}{2}a(v,v) - F(v) \right\}. \tag{2}$$

We can also consider a generalisation of Lax-Milgram:

Theorem 2.17 (Stampacchia). *Let H be a real Hilbert space, $a(u,v)$ a continuous and coercive bilinear form defined on H, and K a non-empty, closed, convex set in H Then, for any $F \in H^*$, there exists $u \in H$ such that*

$$a(u, v-u) \geq F(v-u) \qquad \text{for all } v \in H. \tag{3}$$

Moreover, if $a(\cdot,\cdot)$ is symmetric, then for any u satisfying (3) above

$$\frac{1}{2}a(u,u) - F(u) = \min_{v \in H} \left\{ \frac{1}{2}a(v,v) - F(v) \right\}. \tag{4}$$

Lemma 2.20. *Let H be a real Hilbert space, $M \subset H$ a non-empty, bounded and closed subset, and let $A : M \to M$ be a non-expansive operator. Then, there exists at least one fixed point of the operator A. Moreover, the set of all fixed points is convex.*

This theory holds on Hilbert spaces, we can show that it does not necessarily hold if we weaken to Banach spaces.

Theorem 2.22. *Let B be a closed unit ball in a Hilbert space H and the operator $A : B \to H$ be monotone and continuous on $B \cap M$, where M is an arbitrary finite dimensional subspace of H. Then,*

1. *there exists $x_0 \in B$ such that*

$$(Ax_0, y - x_0) \geq 0 \qquad \text{for all } y \in B.$$

Moreover, the set of points satisfying this condition is convex.

2. *If* $\|x_0\| < 1$ *then* $Ax_0 = 0$. *If* $\|x_0\| = 1$ *and, for each* $x \in \zeta := \{u \in H : \|u\| = 1\}$,

$$x + \lambda Ax \neq 0 \qquad \text{for all } \lambda \geq 0$$

holds. Then,

$$Ax_0 = 0.$$

Theorem 2.23. *Let* $B(w,r)$ *be a closed ball in a Hilbert space* H *with centre at the point* $w \in H$ *and radius* $r > 0$, *and let* $T : B(w,r) \to H$ *be monotone and continuous on* $B(w,r) \cap M$, *where* M *is an arbitrary finite dimensional subspace of* H.

If, for each $z \in S(w,r) := \{u \in H : \|u - w\| = r\}$,

$$z - w + \lambda Tz \neq 0, \qquad \text{for all } \lambda \geq 0,$$

then, there exists $z_0 \in B(w,r)$ *such that* $Tz_0 = 0$.

Exercises

1. Show that Lax-Milgram (Theorem 2.17) can be proven by using Stampacchia (Theorem 2.17)

   Hint. Select $K := H$ and in the inequality (3) select $v = \pm w + u$, where $w \in H$, to get the equality
   $$a(u,w) = F(w) \qquad \text{for } w \in H.$$

2. Prove Theorem 2.17 (Stampacchia).

   Instructions. As in the proof of Lax-Milgram first construct a continuous linear operator $A : H \to H$ such that
   $$a(u,v) = (Au, v) \qquad \text{for each } v \in H.$$
   Then,
   $$\|Au\| \leq c_1\|u\| \qquad \text{and} \qquad (Au, u) \geq c_2\|u\|^2.$$
   Let f be the representation of a continuous linear functional $F \in H^*$; i.e., $F(v) = (v, f)$ for all $v \in H$. Construct a projection P_K from H to K defined by
   $$\|P_K w - w\| = \min_{v \in K}\|w - v\|, \quad w \in H.$$
   From convex set theory it follows this projecction is well defined for closed convex sets, where P_K has the properties
   $$(w - P_K w, v - P_K w) \leq 0 \quad \text{for } v \in K \qquad \text{and} \qquad \|P_K w_1 - P_K w_2\| \leq \|w_1 - w_2\|.$$
   If K is a closed subspace, then P_K is a linear continuous operator for which
   $$(P_K w - w, v) = 0 \qquad \text{for } v \in K.$$
   Show, for any positive $\alpha > 0$, that the following are equivalent:

   - $\alpha(Au, v - u) \geq \alpha(f, v - u)$ for $v \in K$,
   - $(\alpha f - \alpha Au, v - u) \leq 0$ for $v \in K$,
   - $u = P_K(\alpha f - \alpha Au + u)$.

Consider the mapping $S : K \to K$ defined by

$$Sv = P_K(\alpha f - \alpha Av + v), \qquad v \in K.$$

Prove, that for any $x, y \in K$ that

$$\|Sx - Sy\|^2 \leq k^2 \|x - y\|^2, \qquad k^2 = 1 - 2\alpha c_2 + \alpha^2 c_1^2.$$

Choose α such that $\alpha c_1^2 < 2c_2$. Then, $0 < k < 1$, so S is strongly contractive on K and, hence, Banach's fixed point theorem can be applied. Then, the above equivalence gives (3).

If the bilinear form $a(u, v)$ is symmetric, then similarly to the proof of Lax-Milgram, $a(u, v)$ represents an inner product with norm $\|u\|_a$, which you should show is equivalent to the standard norm in H. As from Riesz-Frechét there exists a g such that

$$F(v) = a(g, v), \qquad \text{for all } v \in H,$$

show that

$$a(g - u, v - u) \leq 0, \qquad \text{for all } v \in K.$$

Using Proposition 1.5 show that

$$a(g - u, g - u) = \min_{v \in K} a(g - v, g - v),$$

and from there show that (4) holds.

3. By counterexample, show that Lemma 2.20 does not hold on all Banach spaces, assuming the other assumptions hold.

   Hint. Select $X := c_0$; i.e., the space of numerical sequences converging to 0 with maximum norm:

   $$c_0 := \{x = \{x_i\} : x_i \to 0 \text{ for } i \to \infty\}, \qquad \|x\| = \max_i |x_i|.$$

   Define M as the unit sphere in X and define the operator A as

   $$Ax = \{1, x_1, x_2, \dots\}, \qquad x = \{x_1, x_2, \dots\} \in M.$$

   Clearly M is non-empty, bounded and closed. Show that $A : M \to M$ is non-expansive and, by contradiction, that a fixed point does not exist in M for the operator A.

4. Prove Theorem 2.23.

   Hint. Define the operator $T : B \to H$ as $Ax = T(xr + w)$, $x \in B$ and use Theorem 2.22.