1. Basis $\{e_n\}$, Inner product $(\cdot,\cdot)$, $A_1, A_2 : X \to X \equiv X^*$.

$$A_1 u = -u \qquad A_2 u = \left\{ \begin{array}{ll} u & \text{for } \|u\| \le 1 \\ \frac{u}{\|u\|} & \text{for } \|u\| \ge 1 \end{array} \right\} = \frac{u}{\max(1, \|u\|)}$$

a)

(A_1): let $u_n \to u$, $Au_n \to b$, $\limsup_{n\to\infty} \langle Au_n, u_n \rangle \le \langle b, u \rangle$

$$A_1 u_n \to b \ \& \ A_1 u_n = -u_n \Rightarrow -u_n \to b \Rightarrow u_n \to -b$$

As $u_n \to u$ then $u = -b$

$\therefore A_1 u = -u = b$ as required

(A_2): From Lemma 2.10 monotone & hemicontinuous $\Rightarrow$ pseudomonotone $\Rightarrow (M) \Rightarrow (M)_0$

So just need to show monotone & hemicontinuous

- First show A_2 is monotone:

- Assume $\|u\| \le 1$ & $\|v\| \le 1$

$$(A_2 u - A_2 v, u - v) = (u - v, u - v) = \|u - v\| \ge 0$$

- Assume $\|u\| \ge 1$ & $\|v\| \ge 1$

$$(A_2 u - A_2 v, u - v) = \left(\frac{u}{\|u\|} - \frac{v}{\|v\|}, u - v\right) = \frac{\|u\|^2}{\|u\|} - \frac{(u,v)}{\|u\|} - \frac{(v,u)}{\|v\|} + \frac{\|v\|^2}{\|v\|}$$

$$\ge \frac{\|u\|^2}{\|u\|} - \frac{\|u\|\|v\|}{\|u\|} - \frac{\|v\|\|u\|}{\|v\|} + \frac{\|v\|^2}{\|v\|}$$

$$= 0$$

- Assume $\|u\| \ge 1$ & $\|v\| \le 1$

$$(A_2 u - A_2 v, u - v) = \left(\frac{u}{\|u\|} - v, u - v\right) = \frac{\|u\|^2}{\|u\|} - \frac{(u,v)}{\|u\|} - (v,u) + \|v\|^2$$

$$\ge \frac{\|u\|^2}{\|u\|} - \frac{\|u\|\|v\|}{\|u\|} - \|v\|\|u\| + \|v\|^2$$

$$= \|u\| - \|v\| - \|v\|\|u\| + \|v\|^2$$

$$= \underbrace{(\|u\| - \|v\|)}_{\|v\| \le 1 \le \|u\|} \underbrace{(1 - \|v\|)}_{\ge 0} \ge 0$$

- For $\|u\| \le 1$ & $\|v\| \ge 1$ follows analogously.

Now show continuous $\Rightarrow$ cont. on lines $\Rightarrow$ hemicontinuous (Lemma 2.6)

$$\|Au - Au_n\|^2 = \|Au\|^2 - (Au, Au_n) - (Au_n, Au) + \|Au_n\|^2$$

$$= \left(\frac{\|u\|}{\max(1,\|u\|)}\right)^2 + \underbrace{\left(\frac{\|u_n\|}{\max(1,\|u_n\|)}\right)^2}_{\to \frac{\|u\|}{\max(1,\|u\|)}} - 2\underbrace{\left(\frac{(u,u_n)}{\max(1,\|u\|)\ \max(1,\|u_n\|)}\right)}_{\to \frac{(u,u)}{\max(1,\|u\|)^2}}$$

$$\to 0$$

b) $A = A_1 + A_2 \quad u_n = e_1 + e_n \Rightarrow u_n \rightharpoonup e_1 =: u$

$$Au_n = (A_1 + A_2)u_n = -(e_1 + e_n) + \frac{e_1 + e_n}{\|e_1 + e_n\|}$$
$$= -(e_1 + e_n) + \frac{1}{\sqrt{2}}(e_1 + e_n)$$
$$= \left(\frac{\sqrt{2}}{2} - 1\right)(e_1 + e_n)$$
$$\rightharpoonup e_1\left(\frac{\sqrt{2}}{2} - 1\right) =: b$$

$$\limsup_{n\to\infty}((A_1 + A_2)u_n, u_n) = \limsup_{n\to\infty}\left(\frac{\sqrt{2}}{2} - 1\right)(e_1 + e_n, e_1 + e_n)$$
$$= \sqrt{2} - 2 \leq \frac{\sqrt{2}}{2} - 1 = (u, b)$$

So, LHS of M satisfied, but

$$(A_1 + A_2)u = (A_1 + A_2)e_1 = e_1 - e_1 = 0 \neq b$$

So RHS of M not satisfied

$\Rightarrow$ M not satisfied.

2. X - Separable Hilbert space, orthonormal basis $\{e_n\}$, inner product $(\cdot,\cdot)$

$A: X \to X = X^* \qquad Au = e_1\|u\|, u \in X.$

$\|Au - Av\|^2 = \|Au\|^2 - 2(Au, Av) + \|Av\|^2 = \|u\|^2 - 2\|u\|\|v\| + \|v\|^2 = (\|u\| - \|v\|)^2 \leq \|u - v\|$

$\Rightarrow$ A Lipschitz continuous $\Rightarrow$ A continuous.

- Assume $M \subset X$ bounded.

Need to show $A(M)$ closed & compact. Assume closed for now.

Then any sequence $\{Au_n\} = \{e_1\|u_n\|\} \in A(M)$ has a weakly convergent subsequence because $A(M)$ is a bounded set (M is bounded $\Rightarrow \|u\| \leq C$, for some constant C, $\forall u \in M$

$\Rightarrow \|Au\| = \|e_1\|u\|\| = \|u\|\|e_1\| \leq C \Rightarrow A(M)$ bounded).

Hence, we have a subsequence $\{Au_{n_k}\}$ which converges weakly to v: $(e_1\|u_{n_k}\| - v, w) \to 0 \ \forall w$. As this holds $\forall w$, then $v_1 = Ce_1$ where C is a constant. $\exists u \in X$ st $\|u\| = C$; hence $v = e_1\|u\|$

Therefore $(e_1\|u_{n_k}\| - e_1\|u\|, e_1) \to 0 \Rightarrow \|u_{n_k}\| \to \|u\|$

Now consider $\|Au_{n_k} - Au\|$:

$$\|Au_{n_k} - Au\| = \|Au_{n_k}\|^2 - 2(Au_{n_k}, Au) - \|Au\|$$
$$= \|u_{n_k}\|^2 - 2\|u_{n_k}\|\|u\| + \|u\|^2 \to \|u\|^2 - 2\|u\|^2 + \|u\|^2 = 0$$

$\Rightarrow Au_{n_k} \to Au$

$\Rightarrow$ Every sequence in $A(M)$ has a convergent subsequence

So if $A(M)$ is closed it is compact.

3. $A : X \to X^*$ - real, reflexive Banach space.
$A = B + T$, $B : X \to X^*$ radially continuous, monotone, & bounded
$T : X \to X^*$ strongly continuous

Show A bounded:

$$\|Au\| = \|Bu + Tu\|$$
$$\leq \|Bu\| + \|Tu\|$$

As B is bounded $\exists M_B : \mathbb{R}_+ \to \mathbb{R}_+$ st $\|Bu\| \leq M_B(\|u\|)$

As T strongly continuous $\Rightarrow$ bounded (Lemma 2.6)

$\Rightarrow \exists M_T : \mathbb{R}_+ \to \mathbb{R}_+$ st $\|Tu\| \leq M_T(\|u\|)$

Therefore, $\|Au\| \leq M_B(\|u\|) + M_T(\|u\|) = M_A(\|u\|)$

where $M_A : \mathbb{R}_+ \to \mathbb{R}_+$, $M_A(r) = M_B(r) + M_T(r)$

$\Rightarrow A$ is bounded.

Show A is pseudomonotone:

From Lemma 2.12 the sum of two pseudomonotone operators is pseudomonotone, so just need to show B & T are both pseudomonotone.

- T is strongly continuous $\Rightarrow T$ is pseudomonotone (Lemma 2.10)
- B is monotone, so by Lemma 2.7(4):

B radially continuous $\Leftrightarrow B$ hemicontinuous

B monotone & hemicontinuous $\Rightarrow B$ pseudomonotone (Lemma 2.10) $\square$