Nonlinear Functional Analysis

Practicals

26th March 2020

Notes

1. The theory of monotone operators can be extended to complex Banach (or normed) spaces, where only the real part of the functional is taken into account in the definition of all types of monotonicity (monotone, strictly monotone, strongly monotone, etc.). For example, the operator $A : \mathcal{D} : X \to X^*$, where X is a complex Banach space is called monotone if for all $x, y \in \mathcal{D}(A)$ the inequality
$$\mathrm{Re}\langle Ax - Ay, x - y\rangle \geq 0$$
holds.

2. Monotonicity of an operator $A : \mathcal{D} : X \to X^*$ can also be defined using the notion of a monotone set and the graph of the operator.

   We say that the set $E \subset X \times X^*$ is monotone if for any $(x_1, y_1^*), (x_2, y_2^*) \in E$ the inequality
$$\langle y_1^* - y_2^*, x_1 - x_2\rangle \geq 0$$
is satisfied. Then, A is monotone if its graph $G(A) := \{(x, Ax), x \in \mathcal{D}(A)\}$ is a monotone set.

   A set is *maximally monotone* is its an inclusion-wise maximum set among monotone sets; i.e., it is not a subset of another monotone set. If the graph of the operator is a maximally monotone set then we say that the operator is a *maximally monotone operator*.

3. Let X be a reflexive Banach space. Then all the defined properties (monotonicity, continuity, coercivity, etc., of all types), in Section 2, can be defined for operators of type $B : X^* \to X$ using the canonical mapping $J : X- > X^{**}$, which is linear, isomorphic, and an isometric isomorphism (see Definition 1.2). The operator $B : X^* \to X$ has one of the defined properties if $JB : X^* \to X^{**}$ has the defined property. For example, $B : X^* \to X$ is monotone if the operator $JB : X^* \to X^{**}$ is monotone:
$$\langle y^* - x^*, By^* - Bx^*\rangle = \langle JBy^* - JBx^*, y^* - x^*\rangle \geq 0.$$

Exercises

1. Let X be a separable Hilbert space with a complete orthonormal basis $\{e_n\}$ and inner product $(\cdot, \cdot)$. Define the operators $A_1, A_2 : X \to X \equiv X^*$ as
$$A_1 u = -u,$$
$$A_2 u = \begin{cases} u & \text{for } \|u\| \leq 1, \\ \frac{u}{\|u\|} & \text{for } \|u\| \geq 1. \end{cases}$$

   (a) Verify that A_1 and A_2 satisfy the condition (M).

(b) Show that $A := A_1 + A_2$ does not satisfy (M).

Hint. $u_n = e_1 + e_n$, $u_n \rightharpoonup e_1 = u$.

2. Let X be a separable Hilbert space with a complete orthonormal basis $\{e_n\}$ and inner product $(\cdot,\cdot)$. Define the operator $A : X \to X \equiv X^*$ as $Au = e_1\|u\|$, $u \in X$.

   Show that the operator A is completely continuous but does not satisfy $(M)_0$.

3. Let $A : X \to X^*$, where X is a real reflexive Banach space. Suppose that the decomposition $A = B + T$ exists, where $B : X \to X^*$ is radially continuous, monotone, and bounded and $T : X \to X^*$ is strongly continuous.

   Show that the operator A is pseudomonotone and bounded.