1) Let $A:\mathbb{R}^m \to \mathbb{R}^m$, $m>0$, SPD matrix. Show operator

$$T:\mathbb{R}^m \to \mathbb{R}^m, \quad v \mapsto Av$$

strongly monotone and continuous.

- $\langle x^v, x\rangle$ is standard dot product
- A - symmetric $\Rightarrow A = QDQ^T$, where Q orthogonal & D diagonal $D_{ii} = \lambda_i$

A - SPD: For eigenvalue λ_i, with eigenvector x_i:

$$0 < x^T(Ax) = x^T(\lambda_i x) = \lambda_i x^T x = \lambda_i \|x\|$$

As $\|x_i\| \geq 0 \Rightarrow \lambda_i > 0 \quad \forall \lambda_i \in \sigma(A)$.

So all eigenvalues are strictly positive (hence, can bound sums by min/max).

- Strongly monotone:

$$\begin{aligned}
\langle Au-Av, u-v\rangle &= (Au-Av)^T(u-v) \\
&= (u-v)^T A^T (u-v) \\
&= (u-v)^T Q^T D Q (u-v) \\
&= \sum_{i=1}^{m} ((u-v)^T Q^T)_i \lambda_i (Q(u-v))_i \\
&= \sum_{i=1}^{m} \lambda_i ((u-v)^T Q^T)_i (Q(u-v))_i \\
&\geq \min_i \lambda_i \sum_{i=1}^{n} ((u-v)^T Q^T)_i (Q(u-v))_i \\
&= \underbrace{\min_i \lambda_i}_{=M} (u-v)^T \underbrace{Q^T Q}_{=I} (u-v) \\
&= M\|u-v\|^2
\end{aligned}$$

- Lipschitz Continuous:

$$\begin{aligned}
\langle Au-Av, u-v\rangle &= (Au-Av)^T(u-v) \\
&= (u-v)^T Q^T D Q (u-v) \\
&\leq \underbrace{\max_i \lambda_i}_{=L} (u-v)^T \underbrace{Q^T Q}_{=I} (u-v) \\
&= L\|u-v\|^2
\end{aligned}$$

2). Continuous linear operators are always bounded, whereas continuous nonlinear operators may not be bounded.

For example consider $X := \ell^2$ and define operator $A : X \to X$ as

$$Ax = y \quad x = \{\xi_1, \xi_2, \dots, \xi_k, \dots\}, \quad y = \{(\xi_1)^1, (\xi_2)^2, \dots, (\xi_k)^k \dots\}.$$

A is continuous but not bounded.

Select $x^{(n)} = \{\xi_i^{(n)}\} \in X$ s.t. $x^{(n)} \to x, x = \{\xi_i\}$.

The sequence is bounded, i.e. $\exists C > 0$ s.t.

$\|x^{(n)}\| \leq C$ and $\|x\| \leq C$. From this convergence and definition of the norm in X it follows that there exists N_0 and n_0 such that for all $i \geq N_0$ and $n \geq n_0$ $|\xi_i^{(n)}| < 1/2$ and $|\xi_i| < 1/2$.

Then for $n \geq n_0$

$$\|Ax^{(n)} - Ax\|^2 = \sum_{i=1}^{N_0} \left((\xi_i^{(n)})^i - \xi_i^i\right)^2 + \sum_{i=N_0+1}^{\infty} \left((\xi_i^{(n)})^i - \xi_i^i\right)^2$$

$$\leq C_1 \sum_{i=1}^{N_0} (\xi_i^{(n)} - \xi_i)^2 + C_2 \sum_{i=N_0+1}^{\infty} (\xi_i^{(n)} - \xi_i)^2$$

$$\leq c\|x^{(n)} - x\|^2 \to 0$$

where

$$C_1 = \max_{i=1,\dots,N_0} (iC^{i-1})^2, \quad C_2 = \max_{i=N_0+1,\dots,\infty} (i(1/2)^{i-1})^2$$

$$c = \max(C_1, C_2).$$

which shows continuity.

In the above we used that

$$(a^i - b^i)^2 \leq (a-b)^2 i r^{i-1}$$

for $a \geq 0, b \geq 0, i \in \mathbb{N}, r = \max(a,b)$ and the fact that $\lim_{i \to \infty} (i(\tfrac{1}{2})^{i-1}) \to 0$.

Now show not bounded. Define

$$x^{(n)} = \{\xi_i^{(n)}\}, \quad \xi_n^{(n)} = 2 \text{ and } \xi_i^{(n)} = 0 \text{ for } i \neq n.$$

Then $\|x^{(n)}\| = 2, \quad \|Ax^{(n)}\| = 2^n$

Clearly no function $M_1 : \mathbb{R}_+ \to \mathbb{R}_+$ exists such that

$$\|Au\| \leq M_1(\|u\|),$$

as for such a function to exists

$$\|Ax^{(n)}\| = 2^n \leq M(\|x^{(n)}\|) = M(2) \quad \forall n \in \mathbb{N},$$

which is not possible.