Nonlinear Functional Analysis

Practicals

19th March 2020

1. Let $\boldsymbol{A} \in \mathbb{R}^m \times \mathbb{R}^m$, $m > 0$, be a symmetric positive definite matrix. Show that the operator

$$T : \mathbb{R}^m \to \mathbb{R}^m, \qquad \boldsymbol{v} \mapsto \boldsymbol{A}\boldsymbol{v}$$

is strongly monotone and Lipschitz continuous.

Hint. Consider the eigendecomposition of $\boldsymbol{A}$.

2. Continuous linear operators are *always* bounded; whereas, continuous *nonlinear* operators may not be bounded.

For example, consider $X := \ell^2$ and define the operator $A : X \to X$ as

$$Ax = y, \qquad x = \{\xi_1, \dots, \xi_k, \dots\}, y = \{(\xi_1)^1, \dots, (\xi_k)^k, \dots\}.$$

Show that A is continuous but not bounded.

Hint. For continuity construct a (bounded) convergent sequence. You can also use the trivial statements

$$(a^i - b^i)^2 \le (a - b)^2 i r^{i-1}, \qquad \text{for } a \ge 0, b \ge 0, i \in \mathbb{N}, r = \max(a, b)$$

and

$$\lim_{i \to \infty} \left(i \left(\frac{1}{2} \right)^{i-1} \right) \to 0$$

without proof.