Deadline: 24.11.2021, 11:30

Sobolev embeddings and applications

1. Let $n \geq 2$ and let $B = B(0,1) \subset \mathbb{R}^n$ be the unit ball.
Show that the function $u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$ satisfies $u \in W^{1,n}(B)$ and $u \notin L^\infty(B)$.

Remark. From the Sobolev embedding theorem we know that $W^{1,n}(\Omega) \subset L^q(\Omega)$ for every $q \in [1,\infty)$, provided Ω is a Lipschitz domain. It is therefore a natural question to ask whether also $W^{1,n}(\Omega) \subset L^\infty(\Omega)$. This exercise shows that the answer is negative for $n \geq 2$.

2. Let $u \in W^{1,2}(\mathbb{R}^3)$. Prove that $u \in L^4(\mathbb{R}^3)$ with

$$\|u\|_{L^4(\mathbb{R}^3)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^3)}^{\frac{3}{4}}\|u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}},$$

where the constant $C > 0$ is independent of u.
Hint: Gagliardo–Nirenberg–Sobolev inequality and interpolation of L^p spaces.

Remark. This is a special case of the *Gagliardo–Nirenberg interpolation inequality.*

3. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Suppose that $f_k \rightharpoonup f$ weakly in $L^{\frac{3}{4}}(\Omega)$ and that $u_k \in W_0^{1,2}(\Omega)$ is a weak solution to the problem

$$\begin{aligned} -\Delta u_k + u_k^3 &= f_k \quad \text{in } \Omega \\ u_k &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

that is, $\forall \varphi \in W_0^{1,2}(\Omega) : \int_\Omega \nabla u_k \cdot \nabla\varphi + u_k^3\varphi \,\mathrm{d}x = \int_\Omega f_k\varphi \,\mathrm{d}x.$[1]

- Show that $\int_\Omega |\nabla u_k|^2 + |u_k|^4 \leq C$, where the constant C is independent of k.
- Show that $u_k \to u$ strongly in $L^3(\Omega)$ for some $u \in L^3(\Omega)$.
  Hint: Compact embedding (Rellich–Kondrachov) and interpolation of L^p spaces.
- Show that this u is a weak solution to the problem (note how you use the strong convergence)

$$\begin{aligned} -\Delta u + u^3 &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

[1] This is well defined: we know by the previous exercise that $u_k, \varphi \in L^4$, and moreover $4' = \frac{4}{3}$.