Deadline: 10.11.2021, 11:30

Linear elliptic equations

Let $\Omega \subset \mathbb{R}^n$ be Lipschitz.

1. Prove that the bilinear form

$$(u, v) = \int_\Omega \nabla^2 u : \nabla^2 v \,\mathrm{d}x (= \int_\Omega \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \,\mathrm{d}x), \quad u, v \in W_0^{2,2}(\Omega)$$

is a scalar product on $W_0^{2,2}(\Omega)$.

2. Using the above, show that for any[1] $f \in (W_0^{2,2}(\Omega))^*$ there exists a weak solution $u \in W_0^{2,2}(\Omega)$ to the *biharmonic equation*

$$\begin{aligned} \Delta(\Delta u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} := \nabla u \cdot n &= 0 \quad \text{on } \partial\Omega, \text{ where } n \text{ is the normal to } \partial\Omega. \end{aligned}$$

That is u satisfies

$$(u, v) = \langle f, v \rangle, \quad v \in W_0^{2,2}(\Omega).$$

3. Consider $\mathbb{A} = (a_{ij}^{kl})_{i,j,k,l} \in L^\infty(\Omega, \mathbb{R}^{(n\times n)\times(n\times n)})$ which is uniformly elliptic in Ω, that is there exists $\lambda > 0$ such that for a.e. $x \in \Omega$

$$\forall \boldsymbol{\xi} \in \mathbb{R}^{n\times n}: \quad \mathbb{A}(x)\boldsymbol{\xi} : \boldsymbol{\xi} \geq \lambda |\boldsymbol{\xi}|^2 \quad (\text{i.e. } \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) \xi_{ij} \xi_{kl} \geq \lambda \sum_{i,j=1}^n |\xi_{ij}|^2)$$

Show that for any $f \in (W_0^{2,2}(\Omega))^*$ there exists a weak solution $u \in W_0^{2,2}(\Omega)$ to the equation

$$\begin{aligned} \operatorname{div}\operatorname{div}(\mathbb{A}\nabla^2 u) &= f \quad \text{in } \Omega, \\ (\text{i.e. } \sum_{k,l=1}^n \frac{\partial^2}{\partial x_k \partial x_l} \sum_{i,j=1}^n a_{ij}^{kl} \frac{\partial^2 u}{\partial x_i \partial x_j} &= f \quad \text{in } \Omega) \\ u &= 0 \quad \text{on } \partial\Omega, \\ \mathbb{A}\nabla u \cdot n &= 0 \quad \text{on } \partial\Omega, \text{ where } n \text{ is the normal to } \partial\Omega. \end{aligned}$$

That is, u satisfies

$$(\int_\Omega \sum_{i,j,k,l=1}^n a_{ij}^{kl} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_k \partial x_l} =) \int_\Omega \mathbb{A}\nabla^2 u : \nabla^2 v \,\mathrm{d}x = \langle f, v \rangle, \quad v \in W_0^{2,2}(\Omega).$$

Hint: Modify problem 1. accordingly. Beware that we do not assume $\mathbb{A}$ to be symmetric!

---
[1] if this confuses you, assume $f \in L^2(\Omega)$