Deadline: 20.10.2021, 11:30

Sobolev spaces

1. Prove that the order of weak derivatives can be interchanged.

More precisely: Let $u \in W^{k,p}(\Omega)$ for some $k \in \mathbb{N}$ and $p \in [1, \infty]$. Then for any $i_1, \dots, i_k \in \{1, \dots, n\}$ and any permutation $\pi \colon \{1, \dots, k\} \to \{1, \dots, k\}$ it holds

$$\frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} = \frac{\partial^k u}{\partial x_{i_{\pi(1)}} \cdots \partial x_{i_{\pi(k)}}}.$$

(The order of weak partial derivatives is to be understood in the same way as you're used to, i.e. $\dfrac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} = \dfrac{\partial}{\partial x_{i_1}} \left(\dfrac{\partial^{k-1} u}{\partial x_{i_2} \cdots \partial x_{i_k}} \right)$ etc.)

2. (Poincaré inequality in one dimension) Let $p \in [1, \infty)$ and $0 < L < \infty$. Show that there exists $C \in \mathbb{R}$ such that

$$\forall f \in \mathcal{C}_c^1((0, L)) : \|f\|_p \leq C \|f'\|_p.$$

Try to find the constant C as small as possible![1]

Remark. From the density of $\mathcal{C}_c^1((0, L))$ in $W_0^{1,p}((0, L))$ it follows that the inequality holds for all $f \in W_0^{1,p}((0, L))$. (Check carefully that you understand this argument.)

[1] The smallest possible value of C is sometimes called the *Poincaré constant.*