Deadline: 22.12.2021, 11:30

Linear parabolic equations

Consider the following initial/Neumann boundary value problem for the heat equation, with an unknown function $u\colon [0,T]\times\Omega\to\mathbb{R}$ [1]

$$\begin{aligned} u_t - \Delta u &= f \text{ in } (0,T)\times\Omega \\ \frac{\partial u}{\partial n} &= 0 \text{ on } [0,T]\times\partial\Omega \\ u(0,\cdot) &= g \text{ in } \Omega, \end{aligned}$$

on a given domain $\Omega\subset\mathbb{R}^n$ for a given $f\colon (0,T)\times\Omega\to\mathbb{R}$ and $g\colon\Omega\to\mathbb{R}$, which we assume to be smooth enough.[2]

1. Write the weak formulation for the above problem (note the Neumann boundary values!).

2. Derive a formal a-priori estimate for u by testing the weak formulation with u. That is, assume that u can be taken as a test function and estimate the norm of u (in some suitable spaces) by a constant (which will depend on the data – norms of f and g).

3. Derive a formal a-priori estimate for by testing the weak formulation with u_t. *The same comment as above applies.*

Remark. We do not know that a weak solution u (resp. u_t) is regular enough to be used as a test function. Therefore the estimates obtained above are only*formal*. A way to obtain them rigorously is the following: Perform the Galerkin approximation, then the finite–dimensional solution u^n can be “used as a test function” in the finite–dimensional problem, since testing with u^n is just linear combination of the equation tested with each of the basis functions (the same applies for testing with u^n_t). Then we can pass to the limit.

[1] As usual we denote $u_t := \dfrac{\partial u}{\partial t}$

[2] We will not specify this exactly (for instance we can assume that Ω, f and g are all C^∞) – the assumption is “regular enough that we can do the testing below”.