PARTIAL DIFFERENTIAL EQUATIONS 1

29. 9. 2021

Lebesgue spaces (on $\mathbb{R}^d$)

(repetition)

Let $\Omega \subset \mathbb{R}^d$ be measurable.

<u>Definition</u> (Lebesgue spaces)

Let $p \in [1,\infty]$. Then we define

$L^p(\Omega) = \{ f: \Omega \to \mathbb{R} \text{ measurable} ;\ \|f\|_p < \infty \}$, where

if $p \in [1,\infty)$: $\quad \|f\|_p = \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}$,

if $p = \infty$: $\quad \|f\|_\infty = \operatorname*{essup}_{x \in \Omega} |f(x)| = \inf\{ \alpha > 0 : |f| \leq \alpha \text{ a.e. in } \Omega \}$.

Moreover, for $p = 2$ we define $(f,g)_{L^2} = \int_\Omega fg$.

We identify functions equal a.e., as usual.

Further, let $L^p_{loc}(\Omega) = \{ f: \Omega \to \mathbb{R} \text{ measurable} ;\ f \in L^p(K)\ \forall K \subset \Omega \text{ compact} \}$.

<u>Example</u> $\frac{1}{|x|} \notin L^1((0,1))$, $\frac{1}{|x|} \in L^1_{loc}((0,1))$.

<u>Proposition</u> • $(L^p(\Omega), \|\cdot\|_p)$ is a Banach space, $p \in [1,\infty]$.

• $(L^2(\Omega), (\cdot,\cdot)_{L^2})$ is a Hilbert space.

Example (singularities of L^p functions) Let $p \in [1,\infty)$, $B(0,1) \subset \mathbb{R}^d$ the unit ball, and $f(x) = \frac{1}{|x|^\alpha}$, $x \in \mathbb{R}^d$. Then

(a) $f \in L^p(B(0,1))$ iff $\alpha < \frac{d}{p}$

(b) $f \in L^p(\mathbb{R}^d \setminus B(0,1))$ iff $\alpha > \frac{d}{p}$.

<u>Proof</u> (a)
$$\int_{B(0,1)} |f|^p \underset{\text{integration by spheres}}{=} \int_0^1 \int_{S_r} \frac{1}{|x|^{p\alpha}} \, dS(x)\, dr = \int_0^1 \underbrace{|S_1|}_{\text{area of unit sphere}} \cdot r^{d-1} \cdot \frac{1}{r^{\alpha p}} \, dr = |S_1| \int_0^1 \frac{1}{r^{\alpha p - d + 1}} \, dr < \infty$$

$\overset{\text{Analysis 2}}{\Longleftrightarrow} \alpha p - d + 1 < 1 \Longleftrightarrow \alpha < \frac{d}{p}$

(b) similar

Definition (conjugate exponent) For $p \in [1,\infty]$ define $p' \in [1,\infty]$ such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Lemma (Young's inequality) For any $a, b \geq 0$ and $p \in (1,\infty)$ it holds
$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Proof
$$ab = e^{\log ab} = e^{\frac{1}{p}\log a^p + \frac{1}{p'}\log b^{p'}} \underset{\substack{\uparrow \\ \text{exp convex, } \frac{1}{p}+\frac{1}{p'}=1}}{\leq} \frac{1}{p} e^{\log a^p} + \frac{1}{p'} e^{\log b^{p'}}$$
$\square$

Theorem (Hölder's inequality) Let $f \in L^p(\Omega)$, $g \in L^{p'}(\Omega)$, $p \in [1,\infty]$. Then $fg \in L^1(\Omega)$, $\|fg\|_1 \leq \|f\|_p \|g\|_{p'}$.

Proof Let $\|f\|_p, \|g\|_{p'} > 0$ (otherwise $0 \leq 0$)

$p \in (1,\infty)$: Then for a.e. $x \in \Omega$:
$$\frac{|f(x)||g(x)|}{\|f\|_p \|g\|_{p'}} \overset{\text{Young}}{\leq} \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^{p'}}{p'\|g\|_{p'}^{p'}}$$

integrate over Ω:
$$\frac{\|fg\|_1}{\|f\|_p \|g\|_{p'}} \leq \frac{1}{p} + \frac{1}{p'} = 1$$
$\square$

$p = 1$ ($p = \infty$): $\|fg\|_1 = \int_\Omega |fg| \leq \int_\Omega |f| \|g\|_\infty = \|f\|_1 \|g\|_\infty$.

Corollary (bounded domains) If $|\Omega| < \infty$, $1 \leq p \leq q \leq \infty$, then $L^q(\Omega) \subset L^p(\Omega)$ and $\forall f \in L^q(\Omega)$:
$$\|f\|_p \leq |\Omega|^{\frac{q-p}{pq}} \|f\|_q.$$
[$\frac{1}{p}$ if $q = \infty$]

Proof
$$\|f\|_p = \left\| 1 \cdot |f|^p \right\|_1^{\frac{1}{p}} \overset{\text{Hölder}}{\leq} \|1\|_{(\frac{q}{p})'}^{\frac{1}{p}} \left\| |f|^p \right\|_{\frac{q}{p}}^{\frac{1}{p}} = |\Omega|^{\frac{q-p}{pq}} \|f\|_q$$
$\square$

Corollary (interpolation of L^p spaces) Let $1 \leq p \leq r \leq q \leq \infty$. Then $L^r(\Omega) \subset L^p(\Omega) \cap L^q(\Omega)$. Moreover, for $\theta \in [0,1]$ s.t. $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$
$$\forall f \in L^p(\Omega) \cap L^q(\Omega): \quad \|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta}.$$

Proof Let $\theta \in (0,1)$ (otherwise $r = p$ or $r = q$)
$$\|f\|_r = \left\| |f|^r \right\|_1^{\frac{1}{r}} = \left\| |f|^{\theta r} |f|^{(1-\theta)r} \right\|_1^{\frac{1}{r}} \overset{\text{Hölder}}{\leq}$$
$$\leq \left\| |f|^{\theta r} \right\|_{\frac{p}{\theta r}}^{\frac{1}{r}} \left\| |f|^{(1-\theta)r} \right\|_{\frac{q}{(1-\theta)r}}^{\frac{1}{r}} = \|f\|_p^\theta \|f\|_q^{1-\theta}$$
$\square$

Now we aim to prove density of smooth functions in $L^p(\Omega)$, $1 \le p < \infty$.

Theorem $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$.

<u>Proof</u> Let $f \in L^p(\mathbb{R}^d)$ and $\varepsilon > 0$. By definition of Leboggue integral, we find a simple function $f_2 = \sum_{i=1}^n a_i \chi_{M_i}$, $M_i \subset \mathbb{R}^d$ measurable disjoint, $a_i \in \mathbb{R}$, such that $\|f - f_2\| < \varepsilon$.

Further, by inner regularity of Lebesgue measure, we find compact sets $K_i \subset M_i$, s.t. $\sum_{i=1}^n |M_i \setminus K_i| \le \frac{\varepsilon^p}{n \max_i |a_i|^p}$.

Then $f_3 = \sum_{i=1}^n a_i \chi_{K_i}$ satisfies $\|f_3 - f_2\|_p = \left(\sum |M_i - K_i| \cdot |a_i|^p\right)^{\frac{1}{p}} \le \varepsilon$.

Then $\max_{i \ne j} \operatorname{dist}(K_i, K_j) = d > 0$.

Choose $0 < \delta < \frac{d}{2}$ and put $f_4(x) = \sum_{i=1}^n a_i \max_i \{1 - \frac{1}{\delta} \operatorname{dist}(x, K_i), 0\}$

Then $f_4 \in C_c(\mathbb{R}^d)$ and $f_4 = f_3$ on $K_1, \dots, K_n$, and $f_4 \ne f_3$ only on δ-neighbourhood of K_i, so for δ small we have $\|f_4 - f_3\| < \varepsilon$. Altogether, $\|f - f_4\| < 3\varepsilon$. $\square$

~~Proof idea: it is enough to approximate characteristic functions.~~

~~K compact, $g \in C(\mathbb{R}^d)$~~
~~G open, $g = 1$ on K~~
~~$G \setminus K$ small, $g = 0$ on $\mathbb{R}^d \setminus G$~~
~~$0 \le g \le 1$ on $G \setminus K$~~

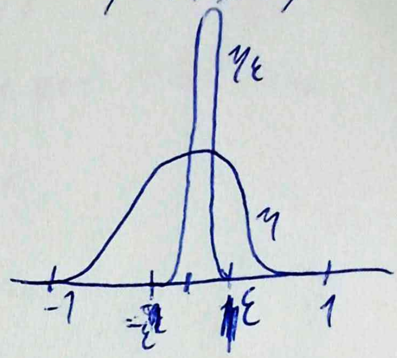
Notation If $|\Omega| < \infty$ (Lebesgue measure), we write $\fint_\Omega f := \frac{1}{|\Omega|} \int_\Omega f$.

Theorem (Lebesgue points) Let $f \in L^1_{loc}(\mathbb{R}^d)$. Then a.e. $x \in \mathbb{R}^d$ is a <u>Lebesgue point</u>, i.e. it satisfies $\lim_{r \to 0^+} \fint_{B(x,r)} |f(y) - f(x)| \, dy = 0$.

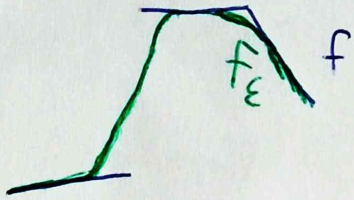
<u>Proof</u> See Real Analysis course (uses Hardy-Littlewood maximal operator).

Definition (Mollification kernel) Let $\eta(x) = \begin{cases} C \cdot e^{\frac{1}{|x|^2-1}}, & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$

where $C = \dfrac{1}{\int_{B(0,1)} e^{\frac{1}{|x|^2-1}}}$. For $\varepsilon > 0$ put $\eta_\varepsilon(x) = \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right)$.



Then
- $\eta \in C_c^\infty(\mathbb{R}^d)$
- $\operatorname{supp} \eta \subset \overline{B(0,1)}$ $\quad (\Rightarrow \operatorname{supp} \eta_\varepsilon \subset \overline{B(0,\varepsilon)})$
- $\eta \geq 0$
- $|x| = |y| \Rightarrow \eta(x) = \eta(y)$
- $\int_{\mathbb{R}^d} \eta = 1$ $\quad (\Rightarrow \int_{\mathbb{R}^d} \eta_\varepsilon = 1)$



Theorem (smoothing of L^p functions). Let $f \in L^1_{loc}(\Omega)$. Define $f = 0$ on $\mathbb{R}^d \setminus \Omega$. Denote $f_\varepsilon := f * \eta_\varepsilon$, ie: $f_\varepsilon(x) = \int_{\mathbb{R}^d} f(y)\eta(x-y)\,dy$, $x \in \mathbb{R}^d$

Then (i) $f_\varepsilon \in C^\infty(\mathbb{R}^d)$

(ii) $f_\varepsilon \to f$ a.e. in Ω.

(iii) $f \in C(G)$, $G \subset \mathbb{R}^d$ open $\Rightarrow f_\varepsilon \overset{loc}{\rightrightarrows} f$ on G. [i.e. $f_\varepsilon \rightrightarrows f$ $\forall K \subset G$ compact]

(iv) $f \in L^p(\Omega) \Rightarrow f_\varepsilon \to f$ in $L^p(\Omega)$.

Proof Notice that after extending by 0 we still have $f \in L^1_{loc}(\mathbb{R}^d)$, so f_ε is well defined.

(i) $f_\varepsilon(x) = \int_{B(x,\varepsilon)} f(y)\eta_\varepsilon(x-y)\,dy$, we can differentiate the integral w.r.t. parameter x_i, we get $\dfrac{\partial f_\varepsilon}{\partial x_i}(x) = \int_{B(x,\varepsilon)} f(y)\dfrac{\partial \eta_\varepsilon}{\partial x_i}(x-y)\,dy$.

We can continue indefinitely.

(ii) Let x be a Lebesgue point of f. Then

$$|f_\varepsilon(x) - f(x)| \leq \int_{B(x,\varepsilon)} |f(x)-f(y)|\,|\eta_\varepsilon(x-y)|\,dy = \frac{1}{\varepsilon^d}\int_{B(x,\varepsilon)} \left|\eta\left(\frac{x-y}{\varepsilon}\right)\right| |f(x)-f(y)|\,dy$$

Hölder

$$\leq |B(x,1)| \cdot \|\eta\|_\infty \fint_{B(x,\varepsilon)} |f(x)-f(y)|\,dy \xrightarrow{\varepsilon \to 0} 0$$

(iii) Let $K \subset G$ compact. Find $K \subset V \subset\subset G$, V open.

$\uparrow$ ($\overline{V}$ compact, $\overline{V} \subset G$.)

Then f is uniformly continuous on $\overline{V}$, from which we get that the limit $\lim\limits_{\varepsilon \to 0} \fint_{B(x,\varepsilon)} |f(x) - f(y)| \, dy = 0$ is uniform in $x \in K$.

Using the same inequality as in (ii) concludes the proof.

(iv) For $f \in L^p(\mathbb{R}^d)$ we first show that $\|f_\varepsilon\|_{L^p(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)}$ $(*)$:

$$\|f_\varepsilon\|_p^p = \int_{\mathbb{R}^d} \|f \cdot \eta_\varepsilon(x - \cdot)\|_1^p \, dx \overset{\text{Hölder}}{\le} \int_{\mathbb{R}^d} \underbrace{\||\eta_\varepsilon(x-\cdot)|^{\frac{1}{p'}}\|_{p'}^p}_{= \|\eta_\varepsilon\|_1^{\frac{p}{p'}} = 1} \cdot \||\eta_\varepsilon(x-\cdot)|^{\frac{1}{p}} \cdot f\|_p^p \, dx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\eta_\varepsilon(x-y)| |f(y)|^p \, dy \, dx \overset{\text{Fubini}}{=} \int_{\mathbb{R}^d} |f(y)|^p \underbrace{\int_{\mathbb{R}^d} |\eta_\varepsilon(x-y)| \, dx}_{=1} \, dy = \|f\|_p^p$$

Now, for $f \in L^p(\mathbb{R}^d)$ find $g \in C_c(\mathbb{R}^d)$, $\|f - g\|_{L^p(\mathbb{R}^d)} < \delta$.

Then
$$\|f - f_\varepsilon\|_{L^p(\mathbb{R}^d)} \le \underbrace{\|f - g\|_{L^p(\mathbb{R}^d)}}_{< \delta} + \underbrace{\|g - g_\varepsilon\|_{L^p(V)}}_{\xrightarrow[\varepsilon \to 0]{} 0} + \underbrace{\|g_\varepsilon - f_\varepsilon\|_{L^p(\mathbb{R}^d)}}_{\overset{(*)}{\le} \|g - f\|_{L^p(\mathbb{R}^d)} < \delta}$$

$V \supset \operatorname{supp} g$ open, $\overline{V}$ compact, $g_\varepsilon \rightrightarrows g$ on V.

$\Rightarrow \limsup\limits_{\varepsilon \to 0} \|f - f_\varepsilon\|_{L^p(\mathbb{R}^d)} < 2\delta \quad \forall \delta > 0.$

$\square$

Remark: This cannot hold for $p = \infty$, because convergence of continuous functions in L^∞ is uniform convergence.

(Thus $\overline{C_c^\infty(\mathbb{R}^d)}^{L^\infty} \subset C(\mathbb{R}^d)$).

Example of a PDE: eigenvalues of the Laplacian

Let $\Omega \subset \mathbb{R}^d$ be open. Recall that for $u: \Omega \to \mathbb{R}$ we have the

<u>Laplace operator</u>
$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}.$$

We have the Laplace equation
$$\Delta u = 0 \quad \text{in } \Omega$$

or the <u>Poisson equation</u>
$$-\Delta u = f \quad \text{in } \Omega, \quad \text{for } f \text{ given.}$$

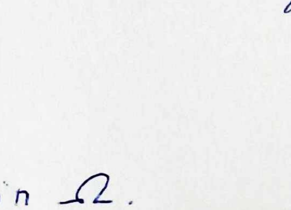
We can also prescribe boundary conditions, such as
$$u = 0 \quad \text{on } \partial\Omega \quad \text{(Dirichlet b.c.)}$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \quad \text{(Neumann b.c.)}$$
normal derivative - normal to $\partial\Omega$ has to be defined

and if Ω is unbounded, some growth condition at ∞.

General principle: $\left.\begin{array}{l}\text{equation in } \Omega \\ \text{data on } \partial\Omega\end{array}\right\} \Rightarrow$ solution in Ω.

Classical theory of PDE: explicit solutions only in special cases
(i.e. Laplace on ball, half space, entire space, ...)
HW 1

<u>Example</u> Let us solve the eigenvalue problem for Laplace (also called Helmholtz equation) in a rectangle:



$$-\Delta u = \lambda u \quad \text{in } \Omega := [0,\ell] \times [0,m]$$
$$u = 0 \quad \text{on } \partial\Omega$$

Solution by separation of variables:

Suppose that $u(x,y) = X(x) \cdot Y(y)$.

Then the equation becomes $X''Y + XY'' = \lambda XY$

$X(0) = X(\ell) = Y(0) = Y(\ell) = 0$.

Formally compute $\underbrace{\frac{X''}{X}}_{\text{independent of } y} = \underbrace{\lambda - \frac{Y''}{Y}}_{\text{independent of } x} =: \mu \in \mathbb{R}$

$\Rightarrow X'' = \mu X$ ... has nonzero solution only if $\mu = \left(\frac{j\pi}{\ell}\right)^2$, $j \in \mathbb{N}$

$X(0) = X(\ell) = 0$ $\quad X(x) = \sin\left(\frac{j\pi}{\ell} x\right)$.

similarly $Y'' = \left(\lambda - \left(\frac{j\pi}{\ell}\right)^2\right) Y$ $\Rightarrow \lambda - \left(\frac{j\pi}{\ell}\right)^2 = \left(\frac{k\pi}{m}\right)^2$, $k \in \mathbb{N}$

$Y(0) = Y(m) = 0$ $\quad Y(y) = \sin\left(\frac{k\pi}{m} y\right)$

$\Rightarrow$ eigenvalues $\lambda_{jk} = \left(\frac{j\pi}{\ell}\right)^2 + \left(\frac{k\pi}{m}\right)^2$, $j, k \in \mathbb{N}$

corresponding eigenfunctions $\varphi_{jk}(x,y) = \sin\left(\frac{j\pi}{\ell}x\right)\sin\left(\frac{k\pi}{m}y\right)$

Remark $\varphi_{j,k}$ are orthogonal in $L^2(\Omega)$:

$$(\varphi_{j_1k_1}, \varphi_{j_2k_2})_{L^2} = \int_\Omega \varphi_{j_1k_1}(x,y)\,\varphi_{j_2k_2}(x,y)\, d(x,y) = \underbrace{\int_0^\ell \sin\left(\frac{j_1\pi}{\ell}x\right)\sin\left(\frac{j_2\pi}{\ell}x\right)dx}_{=\frac{1}{2}\left(\cos\left(\frac{j_1-j_2}{\ell}x\right) - \cos\left(\frac{j_1+j_2}{\ell}x\right)\right)} \cdot$$

$$\cdot \underbrace{\int_0^m \sin\left(\frac{k_1\pi}{m}y\right)\sin\left(\frac{k_2\pi}{m}\right)dy}_{=\frac{1}{2}\left(\cos\left(\frac{k_1-k_2}{m}y\right) - \cos\left(\frac{k_1+k_2}{m}y\right)\right)} = 0 \text{ if } j_1 \neq j_2 \text{ or } k_1 \neq k_2.$$

This is no coincidence, in fact $\{\varphi_{jk} : j, k \in \mathbb{N}\}$ is an orthogonal basis of $L^2(\Omega)$, since:

Theorem Let H be separable Hilbert space, $K : H \to H$ linear, compact, self adjoint. Then there exists a countable basis of H formed by eigenvectors of K

The "solution operator $\Delta^{-1}: L^2(\Omega) \to L^2(\Omega)$" satisfies the assumptions (we will see later)

<u>Attention!</u> boundary conditions are a part of the operator.

For instance, for ~~Dirichlet~~ Neumann b.c. $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, the eigenfunctions would be $\varphi_{jk} = \cos\left(\frac{j\pi}{\ell} x\right) \overset{\cos}{\cancel{\sin}}\left(\frac{k\pi}{\ell} y\right)$.