# Stejnoměrná konvergence

$f_n \to f$ na $M$ $\iff$ $\forall x \in M: \lim\limits_{n\to\infty} f_n(x) = f(x)$

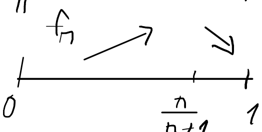
$f_n \rightrightarrows f$ na $M$ $\iff$ $\lim\limits_{n\to\infty} \sigma_n = 0$, kde $\sigma_n = \sup\limits_{x\in M} |f_n(x) - f(x)|$

$\boxed{1}$ $f_n(x) = x^n - x^{n+1}$, $x \in [0,1]$.

**bodová konvergence:** $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^n - x^{n+1} = 0$. Tedy $f_n \to 0$ na $[0,1]$.

**stejnoměrná konvergence:** $\sigma_n = \sup\limits_{x\in[0,1]} |f_n(x) - 0| = \sup\limits_{x\in[0,1]} x^n - x^{n+1} \overset{\Uparrow}{=} f_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) = \underbrace{\left(1 - \frac{1}{n+1}\right)^n}_{\to e^{-1}} \underbrace{\frac{1}{n+1}}_{\to 0}$

$f_n'(x) = n x^{n-1} - (n+1)x^n = \underbrace{x^{n-1}}_{\geq 0}(n - (n+1)x)$



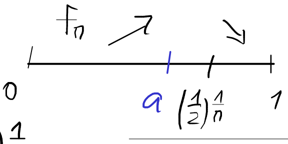

Tedy $\lim\limits_{n\to\infty} \sigma_n = 0$, proto $f_n \rightrightarrows 0$ na $[0,1]$.

$\boxed{2}$ $f_n(x) = x^n - x^{2n}$, $x \in [0,1]$

**bodová konvergence:** $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^n - x^{2n} = 0$ $\Rightarrow f \to 0$ na $[0,1]$

**stejnoměrná konvergence:** $\sigma_n = \sup\limits_{x\in[0,1]} |f_n(x) - 0| = \sup\limits_{x\in[0,1]} f_n(x) \overset{\Uparrow}{=} f_n\left(\left(\frac{1}{2}\right)^{\frac{1}{n}}\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \not\to 0$

$f_n'(x) = n x^{n-1} - 2n x^{2n-1} = n x^{n-1}(1 - 2x^n)$ $\Rightarrow f_n \not\rightrightarrows 0$ na $[0,1]$




pozorování: "problematický bod" $\left(\frac{1}{2}\right)^{\frac{1}{n}} \xrightarrow[n\to\infty]{} 1$

$\Rightarrow$ platí $f_n \overset{loc}{\rightrightarrows} 0$ na $[0,1)$ ?

Podívejme se tedy na stejnoměrnou konvergenci na $[0,a]$, $a<1$.

$\sigma_n = \sup\limits_{x\in[0,a]} |f_n(x) - 0| = \sup\limits_{x\in[0,a]} f_n(x) \overset{\uparrow}{=} f_n(a) = a^n - a^{2n} \xrightarrow[n\to\infty]{} 0$ $\Rightarrow f_n \rightrightarrows 0$ na $[0,a]$

pro $\left(\frac{1}{2}\right)^{\frac{1}{n}} > a$, $f_n$ roste $[0,a]$

Jelikož $a<1$ bylo libovolné, dostáváme $f_n \overset{loc}{\rightrightarrows} 0$ na $[0,1)$.

$\boxed{3}$ $f_n(x) = \dfrac{nx}{1+n+x}$, $x \in [0,1]$.

- $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x}{\underset{\downarrow 0}{\frac{1}{n}} + 1 + \underset{\downarrow 0}{\frac{x}{n}}} = x$. $f(x) = x$, $x \in [0,1]$, $f_n(x) \to f(x)$ na $[0,1]$.

- $\sigma_n = \sup\limits_{x\in[0,1]} |f_n(x) - f(x)| = \sup\limits_{x\in[0,1]} \left| \dfrac{nx}{1+n+x} - x \right| = \sup\limits_{x\in[0,1]} \left| \dfrac{\cancel{nx} - x - \cancel{nx} - x^2}{1+n+x} \right| \leq$

$\leq \dfrac{2}{1+n} \xrightarrow{n\to\infty} 0$ $\Rightarrow f_n \rightrightarrows f$ na $[0,1]$

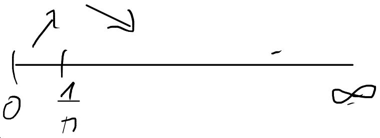
$[\Rightarrow \lim \sigma_n = 0]$

$\boxed{4}$ $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $x \in [0,\infty)$

- $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \underbrace{\dfrac{1}{n}}_{\to 0} \cdot \dfrac{x}{\underbrace{\frac{1}{n^2}+x^2}_{\to \frac{1}{x}}} = 0$, $\quad f_n \to 0$ na $[0,\infty)$

- $\sigma_n = \sup\limits_{x\in[0,\infty)} |f(x) - f_n(x)| = \sup\limits_{x\in[0,\infty)} f_n(x) \overset{\Uparrow}{=} f_n\left(\frac{1}{n}\right) = \dfrac{1}{1+1} = \dfrac{1}{2} \not\to 0 \Rightarrow f_n \not\rightrightarrows 0$ na $[0,\infty)$

$f_n'(x) = \dfrac{n(1+n^2x^2) - nx\cdot 2n^2x}{(1+n^2x^2)^2} = \dfrac{n(1+n^2x^2-2n^2x^2)}{(1+n^2x^2)^2} = \dfrac{n(1-n^2x^2)}{(1+n^2x^2)^2}$



$f_n \overset{loc}{\rightrightarrows} 0$ na $(0,\infty)$? Podívejme se tedy na $[a,b]$, $0<a<b<\infty$

Používáme: $\boxed{f_n \overset{loc}{\rightrightarrows} f \text{ na } I \iff f_n \rightrightarrows f \text{ na } [a,b] \ \forall [a,b] \subset I.}$

$\left( f_n \rightrightarrows 0 \text{ na } [a,\infty) \ ? \right)$ informace navíc / nebudeme řešit

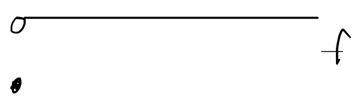
$\sigma_n = \sup\limits_{x\in[a,b]} |f_n(x) - 0| = \sup\limits_{x\in[a,b]} f_n(x) \overset{\uparrow}{=} f_n(a) = \dfrac{na}{1+n^2a^2} \xrightarrow{n\to\infty} 0 \Rightarrow f_n \rightrightarrows 0$ na $[a,b]$

$f_n$ klesá na $[a,b]$ pro $a > \frac{1}{n}$

$[a,b] \subset (0,\infty)$ libovolné $\Rightarrow f_n \overset{loc}{\rightrightarrows} 0$ na $(0,\infty)$.

$\boxed{5}$ $f_n(x) = \arctan(nx)$, $x \in [0,\infty)$

- $\lim\limits_{n\to\infty} \arctan(nx) = \begin{cases} 0, & x = 0 \\ \frac{\pi}{2}, & x \in (0,\infty) \end{cases}$, $\quad f_n \to f = \frac{\pi}{2}\chi_{(0,\infty)}$ na $[0,\infty)$



- Platí věta: $f_n \overset{loc}{\rightrightarrows} f$ na $I \subset \mathbb{R}$, $f_n$ spojité $\Rightarrow f$ spojitá. (viz přednáška)

Máme: $f_n \to f$, $f_n$ spojité, $f$ nespojitá $\Rightarrow f_n \not\rightrightarrows f$ na $[0,\infty)$

Platí $f_n \overset{loc}{\rightrightarrows} f$ na $(0,\infty)$? sami

$\boxed{6}$ $f_n(x) = \dfrac{1}{x+n}$, $x \in [0,\infty)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x+n} = 0, \qquad f_n \to 0, \text{ na } [0,\infty)$$

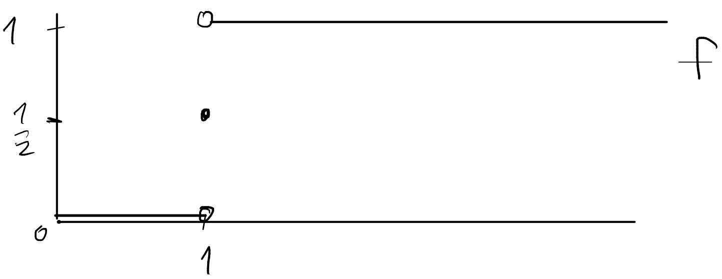
$$\sigma_n = \sup_{x\in[0,\infty)} \frac{1}{x+n} \underset{\uparrow}{=} \frac{1}{0+n} \xrightarrow[n\to\infty]{} 0 \implies f_n \rightrightarrows 0 \text{ na } [0,\infty).$$

výraz klesající v x.

$\boxed{7}$ $f_n(x) = \dfrac{x^n}{1+x^n}$, $x \in [0,\infty)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \left\{ \begin{array}{ll} \lim\limits_{n\to\infty} \dfrac{1}{1+\underbrace{\frac{1}{x^n}}_{\to 0}} = 1, & x > 1 \\ \lim\limits_{n\to\infty} \dfrac{1}{2} = \dfrac{1}{2}, & x = 1 \\ \lim\limits_{n\to\infty} \dfrac{\overbrace{x^n}^{\to 0}}{1+\underbrace{x^n}_{\to 0}} = 0, & 0 \le x < 1. \end{array} \right\} =: f(x)$$

$f_n \to f$ na $[0,\infty)$




$f_n \not\rightrightarrows f$ na $[0,\infty)$, protože $f_n$ spojité, $f$ nespojitá.

$f_n \overset{loc}{\rightrightarrows} f$ na a) $[0,1)$, na b) $(1,\infty)$ ?

Tzn. prověřme stejnoměrnou konvergenci na a) $[0,a]$, $a<1$,

b) $[b,c]$, $1<b<c<\infty$

$$f_n'(x) = \left(\frac{x^n}{1+x^n}\right)' = \frac{nx^{n-1}(1+\cancel{x^n}) - \cancel{x^n \cdot nx^{n-1}}}{(1+x^n)^2} = \frac{nx^{n-1}}{(1+x^n)^2} \ge 0 \text{ na } (0,\infty)$$

a) $\sup\limits_{x\in[0,a]} |f_n(x)-f(x)| = \sup\limits_{x\in[0,a]} f_n(x) = f_n(a) = \dfrac{a^n}{1+a^n} \xrightarrow[n\to\infty]{} 0$

$a<1$ libovolné $\implies f_n \overset{loc}{\rightrightarrows} f$ na $[0,1)$

b) $\sup\limits_{x\in[b,c]} |f_n(x)-f(x)| = \sup\limits_{x\in[b,c]} 1-\dfrac{x^n}{1+x^n} = 1-\dfrac{b^n}{1+b^n} = \dfrac{1}{1+b^n} \xrightarrow[n\to\infty]{} 0$

$1<b<c<\infty$ libovolné $\implies f_n \overset{loc}{\rightrightarrows} f$ na $(1,\infty)$

$\boxed{8}$ $f_n(x) = x \operatorname{arctg}(nx)$, $x \in [0,\infty)$

$$\lim_{n\to\infty} f_n(x) = \left\{ \begin{array}{ll} \lim\limits_{n\to\infty} 0 = 0, & x = 0 \\ \lim\limits_{n\to\infty} x \underbrace{\operatorname{arctg}(\underbrace{nx}_{\to\infty})}_{\to \frac{\pi}{2}} = \frac{\pi}{2} x, & x > 0 \end{array} \right\} = \frac{\pi}{2} x =: f(x)$$

$f_n \to f$ na $[0,\infty)$

$$\sigma_n = \sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} x\left(\frac{\pi}{2} - \operatorname{arctg}(nx)\right) \overset{y = nx}{=} \frac{1}{n} \underbrace{\sup_{y\in[0,\infty)} y\left(\frac{\pi}{2} - \operatorname{arctg} y\right)}_{= C < \infty \text{ (ověř)}}$$

$$= \frac{1}{n}\cdot C \xrightarrow{n\to\infty} 0 \implies f_n \rightrightarrows f \text{ na } [0,\infty).$$

$\boxed{9}$ $f_n(x) = n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right) = \cancel{n}\cdot \dfrac{\cancel{x} + \cancel{\frac{1}{n}} - \cancel{x}}{\sqrt{x+\frac{1}{n}} + \sqrt{x}}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{\sqrt{x+\frac{1}{n}} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = f(x)$$

$f_n \to f$ na $(0,\infty)$

Nebo: všimneme si $\lim\limits_{n\to\infty} f_n(x) \overset{\text{Heine}}{=} \dfrac{d}{dt}\sqrt{t}\Big|_{t=x} = \dfrac{1}{2\sqrt{x}}$

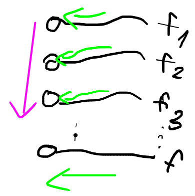
$f_n \rightrightarrows f$ na $(0,\infty)$ ?

$$\sigma_n = \sup_{x\in(0,\infty)} \left| \frac{1}{\sqrt{x+\frac{1}{n}} + \sqrt{x}} - \frac{1}{2\sqrt{x}} \right| = \sup_{x\in(0,\infty)} \frac{-\cancel{2}\sqrt{x} + \sqrt{x+\frac{1}{n}} - \cancel{\sqrt{x}}}{2\sqrt{x}\left(\sqrt{x+\frac{1}{n}} + \sqrt{x}\right)} =$$

$$= \sup_{x\in(0,\infty)} \frac{\cancel{x} + \frac{1}{n} - \cancel{x}}{2\sqrt{x}\left(\sqrt{x+\frac{1}{n}} + \sqrt{x}\right)^2} \geq \lim_{x\to 0+} \frac{\frac{1}{n}}{2\underbrace{\sqrt{x}}_{\to 0}\underbrace{\left(\sqrt{x+\frac{1}{n}} + \sqrt{x}\right)^2}_{\to \frac{1}{n}}} = \infty \implies f_n \not\rightrightarrows f \text{ na } (0,\infty)$$

alternativní postup: Moore-Osgood

$$\left.\begin{array}{l} \text{(i) } f_n \rightrightarrows f \text{ na } (0,\varepsilon) \\ \text{(ii) } \exists \lim\limits_{x\to 0+} f_n(x) \in \mathbb{R} \end{array}\right\} \implies \lim_{x\to 0+} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to 0+} f_n(x) \in \mathbb{R}$$




náš příklad: (ii) $\lim\limits_{x\to 0+} f_n(x) = \dfrac{1}{\sqrt{0+\frac{1}{n}} + \sqrt{0}} = \sqrt{n}$ ✓

$$\lim_{x\to 0+} \lim_{n\to\infty} f_n(x) = \lim_{x\to 0+} \frac{1}{2\sqrt{x}} = \infty \notin \mathbb{R}$$

závěr M-O věty neplatí

$\implies$ neplatí (i)

Tedy $f_n \not\rightrightarrows f$ na $(0,\varepsilon)$

$f_n \overset{loc}{\rightrightarrows} f$ na $(0,\infty)$?

$$\sigma_n = \sup_{x\in[a,b]} |f_n(x)-f(x)| = \dots \qquad [a,b]\subset(0,\infty)$$

alternativní postup: Dini

$$f_n \longrightarrow f \text{ na } K \implies f_n \rightrightarrows f \text{ na } K$$

spojité  monotonní ($f_n(x)\le f_{n+1}(x)$ nebo $\ge$)  spojitá  kompakt.

náš příklad: $[a,b]\subset(0,\infty)$ kompakt. ✓

$f_n, f$ spoj ✓

$$f_n(x) = \frac{1}{\sqrt{x+\frac{1}{n}}+\sqrt{x}} \quad \text{monotonní v } n \ ✓$$

Dini $\implies f_n \rightrightarrows f$ na $[a,b]$

$[a,b]\subset(0,\infty)$ libovolné $\implies f_n \overset{loc}{\rightrightarrows} f$ na $(0,\infty)$

poznámka (spíš jen pro zajímavost):

platí: $g'$ stejnoměrně spojitá $\implies \dfrac{g(x+\frac{1}{n})-g(x)}{\frac{1}{n}} \rightrightarrows g'$

náš příklad: $g(x)=\sqrt{x}$, pak $f_n = \dfrac{g(x+\frac{1}{n})-g(x)}{\frac{1}{n}}$

$g'(x)=\dfrac{1}{2\sqrt{x}}$, stejn. spoj. na $[a,b]\subset(0,\infty)$ $\implies f_n \rightrightarrows g'$ na $[a,b]$

$\boxed{11}$ $f_n(x) = \sqrt[n]{x^n+3^n}$, $x\in[0,\infty)$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} x, & x\ge 3 \\ 3, & x<3 \end{cases}$$

$x \ge 3$: $\sqrt[n]{x^n} \le f_n(x) \le \sqrt[n]{2x^n}$, $\nearrow x$, $\nearrow x$

$x < 3$: $\sqrt[n]{3^n} \le f_n(x) \le \sqrt[n]{2\cdot 3^n}$, $3$, $\searrow 3$

+ strážníci

$f_n \rightrightarrows f$ na $[0,3)$?

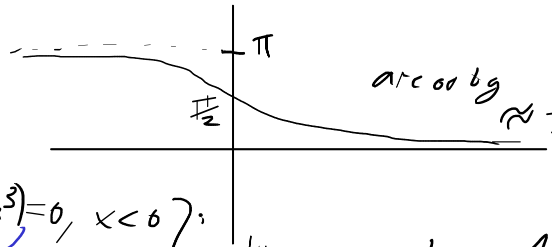
$$\sigma_n = \sup_{x\in[0,3)} \sqrt[n]{x^n+3^n}-3 = \sup_{x\in[0,3)} \frac{x^n}{\sum_{k=0}^{n-1}(x^n+3^n)^{\frac{k}{n}}\cdot 3^{n-k-1}} = \sup_{x\in[0,3)} \underbrace{\frac{x^n}{3^n}}_{\le 1} \frac{3}{\underbrace{\sum_{k=0}^{n-1}\Big(\underbrace{\big(\tfrac{x}{3}\big)^n}_{\ge 0}+1\Big)^{\frac{k}{n}}}_{\ge 1}}$$

$$\le 1\cdot\frac{3}{n} \xrightarrow{n\to\infty} 0 \implies \sigma_n \xrightarrow[n\to\infty]{} 0 \implies f_n \rightrightarrows f \text{ na } [0,3)$$

$f \rightrightarrows f$ na $[3,\infty)$?

$$\sup_{x\in[3,\infty)} \sqrt[n]{x^n+3^n}-x \ \dots \ \text{analogicky} \implies f_n \rightrightarrows f \text{ na } [3,\infty)$$

$\boxed{12}$ $f_n(x) = (x+1)^3 \operatorname{arccotg}(-nx^3)$




$\operatorname{arccotg} \approx \frac{1}{x}$

$$\lim_{n\to\infty} f_n(x) = \left\{ \begin{array}{ll} \lim\limits_{n\to\infty} \underbrace{-\frac{1}{n}}_{\to 0}\underbrace{\frac{(x+1)^3}{x^3}}_{\in\mathbb{R}} \underbrace{(-nx^3)\operatorname{arccotg}(-nx^3)}_{\to 1} = 0, & x<0 \\ \frac{\pi}{2} & x=0 \\ \lim\limits_{n\to\infty} (x+1)^3 \underbrace{\operatorname{arccotg}(-nx^3)}_{\to\pi} = \pi(x+1)^3 & \end{array} \right\} f(x)$$

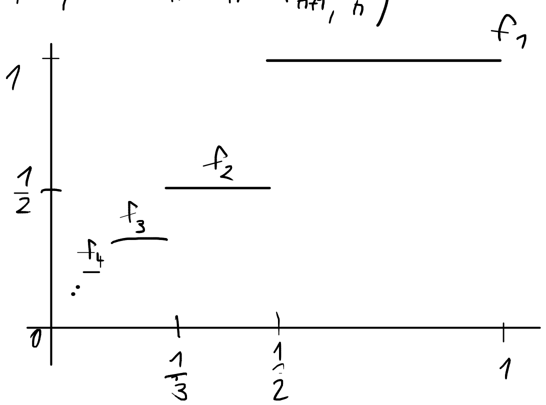
$$\lim_{x\to\infty} \frac{\operatorname{arccotg} x}{\frac{1}{x}} = 1$$

$f_n \to f$ na $\mathbb{R}$.

# Stejnoměrná konvergence řad

Weierstrass: $S_n = \sup_{x\in M} |f_n(x)|$, $\sum_{n=1}^{\infty} S_n < \infty \Rightarrow \sum_{n=1}^{\infty} f_n \rightrightarrows$ na $M$.

$\cancel{\Leftarrow}$, např.: $f_n = \frac{1}{n}\chi_{(\frac{1}{n+1}, \frac{1}{n})}$




Pak: $S_n = \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$,

ale $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \sup_{x\in[0,1]} \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{\infty} f_k(x) \right| =$

$= \lim_{n\to\infty} \sup_{x\in[0,1]} \left| \sum_{k=n+1}^{\infty} f_k(x) \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\Rightarrow \sum_{n=1}^{\infty} f_n \rightrightarrows$ na $[0,1]$

$\boxed{1}$ $\sum_{n=1}^{\infty} \underbrace{\frac{\cos(nx)}{n^2}}_{f_n(x)}$, $x \in \mathbb{R}$

$S_n = \sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \frac{|\cos nx|}{n^2} = \frac{1}{n^2}$, $\sum_{n=1}^{\infty} S_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \xRightarrow{\text{Weierstrass}} \sum_{n=1}^{\infty} f_n \rightrightarrows$ na $\mathbb{R}$.

Pokud označíme $f(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$, $x\in\mathbb{R}$, pak máme $\underbrace{\sum_{k=1}^{n} f_k}_{\text{spojitá}} \rightrightarrows f$ na $\mathbb{R}$, tedy $f$ je spojitá na $\mathbb{R}$.

$\boxed{2}$ $\sum_{n=0}^{\infty} \underbrace{x^n}_{f_n(x)}$

Víme: konverguje pro $x\in(-1,1)$, a platí $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. $\rightsquigarrow$ $f$ je spojitá na $(-1,1)$.

$\sum_{n=0}^{\infty} f_n \rightrightarrows$ na $(-1,1)$? Spočteme $S_n = \sup_{x\in(-1,1)} |x^n| = 1$, $\sum_{n=0}^{\infty} 1 = \infty \Rightarrow$ nevíme nic.

Moore-Osgood: Kdyby $\sum f_n \rightrightarrows f$ na $(-1,1)$, platilo by $\sum_{n=0}^{\infty} \lim_{x\to 1^-} x^n = \underbrace{\lim_{x\to 1^-} \sum_{n=0}^{\infty} x^n}$ $\in \mathbb{R}$

Ale víme, že $\lim_{x\to 1^-} \frac{1}{1-x} = \infty$, tedy $\sum_{n=0}^{\infty} f_n \not\rightrightarrows$ na $(-1,1)$.

$\sum_{n=0}^{\infty} f_n \overset{loc}{\rightrightarrows}$ na $(-1,1)$?

Podíváme se na $[-a,a]$, $0<a<1$.

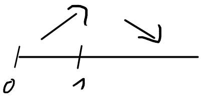
Pak $S_n = \sup_{x\in[-a,a]} |x^n| = a^n$, $\sum_{n=0}^{\infty} S_n = \sum_{n=0}^{\infty} a^n \in \mathbb{R} \xRightarrow{\text{Weierstrass}} \sum_{n=0}^{\infty} f_n \rightrightarrows$ na $[-a,a]$

tedy $\overset{loc}{\rightrightarrows}$ na $(-1,1)$

Poznámka: jde o speciální případ mocninné řady.

$\boxed{3}$ $\sum\limits_{n=1}^{\infty} \underbrace{x^n e^{-nx}}_{f_n(x)}$ , $x \in [0,\infty)$

$(x^n e^{-nx})' = n x^{n-1} e^{-nx} - n x^n e^{-nx} =$
$= n x^{n-1} e^{-nx}(1-x)$ , $f_n \geq 0$
$\Rightarrow$ maximum v 1




$S_n = \sup\limits_{x\in[0,\infty)} |f_n(x)| = \sup\limits_{x\in[0,\infty)} x^n e^{-nx} = 1^n e^{-n\cdot 1} = e^{-n}$

$\sum\limits_{n=1}^{\infty} S_n = \sum\limits_{n=1}^{\infty} e^{-n} \in \mathbb{R} \Rightarrow \sum\limits_{n=1}^{\infty} f_n \rightrightarrows$ na $[0,\infty)$

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$\boxed{4}$ $\sum\limits_{n=1}^{\infty} \underbrace{e^{-(nx+\frac{1}{nx})}}_{f_n(x)}$ , $x \neq 0$ , bodová konvergence $(0,\infty)$: $|f_n(x)| \leq e^{-nx}$,
$\sum\limits_{n=1}^{\infty} e^{-nx} \in \mathbb{R} \Rightarrow \sum\limits_{n=1}^{\infty} f_n(x) \in \mathbb{R}$

Pro $x<0$: $f_n(x) \geq e^{-nx} \xrightarrow{n\to\infty} \infty$, tedy $\sum\limits_{n=1}^{\infty} f_n$ diverguje

Pro $x \in (0,\infty)$ Všimneme si: $f_n\left(\frac{1}{n}\right) = e^{-n\cdot\frac{1}{n} - \frac{1}{n\frac{1}{n}}} = e^{-2}$.

[Chceme $f_n \not\rightrightarrows 0$] Pak máme $\sigma_n = \sup\limits_{x\in[0,\infty)} |f_n(x) - 0| \geq |f_n(\tfrac{1}{n}) - 0| = e^{-2} \underset{n\to\infty}{\not\to} 0$

Tedy $f_n \not\rightrightarrows 0$.

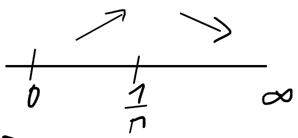
Tedy není splněna nutná podmínka stejnoměrné konvergence, tedy
$[\sum f_n \rightrightarrows \;\Rightarrow\; f_n \rightrightarrows 0]$
$\sum\limits_{n=1}^{\infty} f_n \not\rightrightarrows$ na $(0,\infty)$.

lokálně stejnoměrná na $(0,\infty)$:

Vezmeme interval $[a,b] \subset (0,\infty)$.

$S_n = \sup\limits_{x\in[a,b]} |f_n(x)| =$

$f_n'(x) = e^{-(nx+\frac{1}{nx})} \cdot \left(-\left(n - \frac{1}{nx^2}\right)\right) = e^{-(nx+\frac{1}{nx})}\left(\frac{1}{nx^2} - n\right)$




$\overset{\uparrow}{=} f_n(a)$ , Víme že $\sum\limits_{n=1}^{\infty} f_n(a) \in \mathbb{R}$
$\Downarrow$
Pro $n$ t. ž. $a > \frac{1}{n}$ $\quad \sum f_n \rightrightarrows$ na $[a,b]$ , tedy $\sum\limits_{n=1}^{\infty} f_n \overset{loc}{\rightrightarrows} (0,\infty)$.

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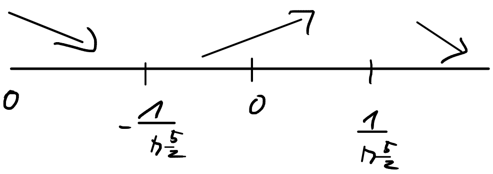
$\boxed{5}$ $\sum\limits_{n=1}^{\infty} \underbrace{\frac{nx}{1+n^5x^2}}_{f_n(x)}$ , $x \in \mathbb{R}$

$f_n'(x) = \frac{n(1+n^5x^2) - nx\,2xn^5}{(1+n^5x^2)^2} = \frac{n - n^6x^2}{(1+n^5x^2)^2} =$
$= \frac{n(1-n^{\frac{5}{2}}x)(1+n^{\frac{5}{2}}x)}{(1+n^5x^2)^2}$

$S_n = \sup\limits_{x\in\mathbb{R}} |f_n(x)| = \frac{n^{-\frac{3}{2}}}{2}$.

Platí $\sum\limits_{n=1}^{\infty} \frac{n^{-\frac{3}{2}}}{2} \in \mathbb{R}$

Weierstrass $\Rightarrow \sum\limits_{n=1}^{\infty} f_n \rightrightarrows$ na $\mathbb{R}$.




$\lim\limits_{x\to\pm\infty} f_n(x) = \lim\limits_{n\to\pm\infty} \frac{n\cancel{x}}{1+n^5x^{\cancel{2}}} = 0$

$f_n\left(\frac{1}{n^{\frac{5}{2}}}\right) = \frac{n^{-\frac{3}{2}}}{1+n^5n^{-5}} = \frac{n^{-\frac{3}{2}}}{2}$

$f_n\left(-\frac{1}{n^{\frac{5}{2}}}\right) = -\frac{n^{-\frac{3}{2}}}{2}$

$\boxed{6}$ $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{2x}{x^2+n^2}}_{=f_n(x)}$, $x \in \mathbb{R}$

<u>na $[-a,a]$, $a>0$:</u> $\displaystyle\left|\frac{2x}{x^2+n^2}\right| \leq \frac{2a}{n^2}$, $\displaystyle\sum_{n=1}^{\infty}\frac{2a}{n^2} \in \mathbb{R} \overset{\text{Weierstrass}}{\Longrightarrow} \sum f_n \rightrightarrows$ na $[-a,a]$

$\Rightarrow \sum f_n \overset{loc}{\rightrightarrows}$ na $\mathbb{R}$

$\sup_{x\in\mathbb{R}} g(x) \geq g(N)$

<u>na celém $\mathbb{R}$:</u> $\displaystyle\sup_{x\in\mathbb{R}}\sum_{n=N}^{2N}\frac{2x}{x^2+n^2} \geq \sup_{x\in\mathbb{R}}\sum_{n=N}^{2N}\frac{2x}{0+(2N)^2} \geq \sum_{n=N}^{2N}\frac{2N}{4N^2} = \frac{1}{2}$

není splněna BC podmínka, tedy $\sum f_n \not\rightrightarrows$ na $\mathbb{R}$.

$\boxed{6}$ $\displaystyle\sum_{n=2}^{\infty} \underbrace{\log\left(1+\frac{x^2}{n\log^2 n}\right)}_{f_n(x)}$

Např. $\displaystyle\int_2^{\infty}\frac{a^2}{t\log^2 t}\,dt = \int_{\log 2}^{\infty}\frac{a^2}{z^2}\,dz < \infty$ + integrální kritérium konvergence řad.

$\displaystyle|f_n(x)| \leq \frac{x^2}{n\log^2 n} \overset{\text{pro } x\in[-a,a]}{\leq} \frac{a^2}{n\log^2 n}$, $\displaystyle\sum_{n=2}^{\infty}\frac{a^2}{n\log^2 n} \in \mathbb{R} \Rightarrow \sum f_n \rightrightarrows$ na $[-a,a]$

$\log t \leq t-1$, $t \in (0,\infty)$

$\Rightarrow \sum f_n \overset{loc}{\rightrightarrows}$ na $\mathbb{R}$.

$\boxed{7}$ úloha 5 a $t-1 \geq \log t$, $t \in (0,\infty)$.

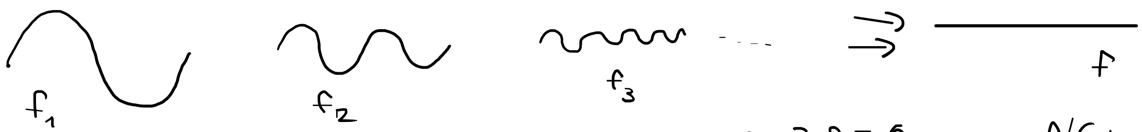
# Stejnoměrná konvergence a derivace

$f_n:(a,b)\to\mathbb{R}$, • $f_n' \rightrightarrows$ na $(a,b)$, • $\exists x_0\in(a,b): \lim\limits_{n\to\infty} f_n(x_0)\in\mathbb{R}$ $\Big\}\Rightarrow$ $f_n\rightrightarrows f$, $f_n'\rightrightarrows f'$.

$\left[\text{tzn. } \left(\lim\limits_{n\to\infty} f_n(x)\right)' = \lim\limits_{n\to\infty} f_n'(x)\right]$.

Poznámka: $f_n\rightrightarrows f$ nestačí:

$f_n(x)=\frac{1}{n}\sin(nx)$




$f_n'(x)=\cos(nx)$ ... nekonverguje bodově, přestože $f_n\rightrightarrows f\equiv 0$ a $f'(x)=0$.

Věta pro řady: • $\sum f_n' \rightrightarrows$ na $(a,b)$ • $\exists x_0\in(a,b): \sum f_n(x_0)\in\mathbb{R}$ $\Big\}\Rightarrow$ $\sum f_n' \rightrightarrows$, $\left(\sum f_n\right)'=\sum f_n'$

$\boxed{9}$ $f(x)=\sum\limits_{n=0}^{\infty}\underbrace{\frac{x^n}{n!}}_{f_n(x)}$, $x\in\mathbb{R}$, $\overset{?}{\Rightarrow}$ $f$ řeší $y'=y$, $y(0)=1$.

Ukážeme: $\sum\limits_{n=1}^{\infty}\frac{x^n}{n!}\overset{loc}{\rightrightarrows}$ na $\mathbb{R}$:

na $[-a,a]$: $\left|\frac{x^n}{n!}\right|\le\frac{a^n}{n!}$, $\sum\frac{a^n}{n!}\in\mathbb{R}\overset{\text{Weierstrass}}{\Rightarrow}\sum\rightrightarrows$ na $[-a,a]$

$\Rightarrow \sum f_n \overset{loc}{\rightrightarrows}$ na $\mathbb{R}$ $\Rightarrow$ $f$ je spojitá na $\mathbb{R}$.

konvergence derivací?

$f_n'(x)=\left(\frac{x^n}{n!}\right)'=\frac{nx^{n-1}}{n!}=\begin{cases}\frac{x^{n-1}}{(n-1)!}, & n\ge 1.\\ 0, & n=0\end{cases}$

tedy $\sum\limits_{n=0}^{\infty} f_n'(x)=\sum\limits_{n=\cancel{0}}^{\infty}\frac{x^{n-1}}{(n-1)!}=\sum\limits_{k=0}^{\infty}\frac{x^k}{k!}\overset{loc}{\rightrightarrows}$ na $\mathbb{R}$

víme viz výše.

Tedy platí $f'(x)=\sum\limits_{n=0}^{\infty}f_n'(x)=\sum\limits_{k=0}^{\infty}\frac{x^k}{k!}=f(x)$, $x\in\mathbb{R}$. $\rightsquigarrow$ $f$ řeší $y'=y$ na $\mathbb{R}$.

$f(0)=1+0+0+\dots=1$.

$\boxed{10}$ $f(x)=\sum\limits_{n=1}^{\infty}\underbrace{(-1)^n\frac{\sin\left(1+\frac{x}{n}\right)}{\sqrt{n}}}_{f_n(x)}$, $f'(0)=?$ $[\exists f'(0)?]$

$|f_n'(x)|=\left|(-1)^n\frac{\cos\left(1+\frac{x}{n}\right)\cdot\frac{1}{n}}{\sqrt{n}}\right|\le\frac{1}{n^{\frac{3}{2}}}$, $x\in(-1,1)$.

$\sum\limits_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}<\infty\overset{\text{Weierstrass}}{\Rightarrow}$ • $\sum f_n'\rightrightarrows$ na $(-1,1)$

• $x_0=0$: $\sum\limits_{n=1}^{\infty}(-1)^n\frac{\sin 1}{\sqrt{n}}\overset{\text{Leibniz}}{\in}\mathbb{R}$ $\Big\}\Rightarrow$ $f'(x)=\sum\limits_{n=1}^{\infty}f_n'(x)$ na $(-1,1)$

Spec. pro $x=0$: $f'(0)=\sum\limits_{n=1}^{\infty}\frac{(-1)^n\cos 1}{n^{\frac{3}{2}}}$.

$\boxed{11}$ $\sum_{n=1}^{\infty} \underbrace{\frac{\sin(nx)}{n}}_{=f_n(x)}$, $x \in (0,\pi)$.

Pozn.: Abelovo a Dirichletovo kriterium na přednášce na přednášce letos nebylo, berte tedy tedy tyto úlohy spíše informativně

stejnoměrná na $(0,\pi)$: ukážeme, že není splněna BC podmínka

$$\sup_{x\in(0,\pi)} \sum_{n=N}^{2N} \frac{\sin(nx)}{n} \geq \sum_{n=N}^{2N} \frac{\sin(n\cdot\frac{1}{N})}{n} \geq \sin 1 \cdot \sum_{n=N}^{2N} \frac{1}{n} \geq \sin 1 \cdot \sum_{n=N}^{2N} \frac{1}{2N} = \sin 1 \cdot N \cdot \frac{1}{2N} = \frac{\sin 1}{2} > 0$$

$1 \leq \frac{n}{N} \leq 2 \Rightarrow \sin\frac{n}{N} \geq \sin 1$

$\Rightarrow \sum f_n \not\rightrightarrows$ na $(0,\pi)$.

lokálně stejnoměrná:

na $[\delta, \pi-\delta]$: Dirichlet: $\sum f_n g_n \rightrightarrows$

nástejně omez. částečné součty; $g_n \rightrightarrows 0$, $g_n$ monotonní v $n$.

Položme $g_n(x) := \frac{1}{n}$, $f_n(x) = \sin(nx)$ ... má omez. č. s? ✓

$g_n$ klesá do 0 ✓

Platí $\left|\sum_{n=1}^{N} \sin(nx)\right| = \left|\frac{\cos\frac{x}{2} - \cos((N+\frac{1}{2})x)}{2\sin\frac{x}{2}}\right| \leq \frac{2}{2\sin\frac{\delta}{2}} = \frac{1}{\sin\frac{\delta}{2}}$

[ Dk: $\cos((n-\frac{1}{2})x) - \cos((n+\frac{1}{2})x) = \cos nx \cancel{\cos(-\frac{1}{2}x)} - \sin nx \sin(-\frac{1}{2}x) - \cos nx \cancel{\cos(\frac{1}{2}x)} + \sin nx \sin(\frac{1}{2}x) = 2\sin nx \sin\frac{x}{2}$ ]

Vyděl $2\sin\frac{x}{2}$, sečti pro $n = 1,\dots,N$

Dirichlet $\Rightarrow \sum f_n g_n = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \rightrightarrows$ na $[\delta, \pi-\delta]$,

Tedy $\sum \frac{\sin nx}{n} \overset{loc}{\rightrightarrows}$ na $(0,\pi)$.

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$\boxed{12}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+x}$, $x \in (-1,\infty)$

$f_n(x) = (-1)^n$, $g_n(x) = \frac{1}{n+x}$

stejně omez. část. součty

$g_n \rightrightarrows 0$ (viz minule)

$g_n(x)$ monotonní v $n$.

Dirichlet $\Rightarrow \sum \rightrightarrows$ na $(-1,\infty)$.

$g_n \overset{?}{\rightrightarrows} 0$ na $(-1,\infty)$ $\sigma_n = \sup_{(-1,\infty)} g_n = \begin{cases} \infty, & n=1 \\ \frac{1}{n-1}, & n>1 \end{cases}$ $\xrightarrow{n\to\infty} 0$

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$\boxed{13}$ $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n(1+x^n)}$, $x \in (0,1)$

Abel: $\sum f_n g_n \rightrightarrows$

$\sum f_n \rightrightarrows$

$g_n$ monotonní v $n$ a stejně omezená

$f_n(x) = \frac{(-1)^n}{n}$, $g_n(x) = \frac{x^n}{1+x^n}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \in \mathbb{R} \Rightarrow \sum_{n=1}^{\infty} f_n \rightrightarrows$,

$g_n(x) = 1 - \frac{1}{1+x^n}$ je monotonní v $n$.

$0 \leq g_n(x) \leq 1$ ... $g_n$ je stejně omez.

$\xrightarrow{\text{Abel}} \sum \rightrightarrows$ na $(0,1)$.

$\boxed{17}$ $f(x) = \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n}{n+x}}_{f_n(x)}$, $x \in (-1, \infty)$. Existence $f'$?

- $\underline{x_0 := 0}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \in \mathbb{R}$ Leibniz
- $f_n'(x) = \frac{(-1)^{n+1}}{(n+x)^2}$.

Výraz klesající v $x$, $n > 1$.

$S_n = \sup_{x \in (-1,\infty)} |f_n'(x)| = \sup_{x \in (-1,\infty)} \frac{1}{(n+x)^2} \doteq \frac{1}{(n-1)^2}$

$\sum_{n=1}^{\infty} \frac{1}{(n-1)^2} \in \mathbb{R} \overset{\text{Weierstrass}}{\Longrightarrow} \sum f_n' \rightrightarrows$ na $(-1,\infty)$.

$\Longrightarrow$ $f'(x) = \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+x)^2}$, $x \in (-1, \infty)$.

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$\boxed{18}$ $f(x) = \sum_{n=1}^{\infty} \frac{|x|}{n^2+x^2}$, $x \in \mathbb{R}$.

Pišme $g(x) = \sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2+x^2}}_{g_n(x)}$, pak $f(x) = |x| \cdot g(x)$.

- $\underline{x_0 = 0}$: $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2} \in \mathbb{R}$ ✓
- $g_n'(x) = \frac{-2x}{(n^2+x^2)^2}$.

Pak pro $x \in [-a,a]$: $|g_n'(x)| \leq \frac{2a}{n^4}$,

$\sum_{n=1}^{\infty} \frac{2a}{n^4} \in \mathbb{R} \overset{\text{Weierstrass}}{\Longrightarrow} \sum g_n' \rightrightarrows$ na $[-a,a]$

$\Longrightarrow \sum g_n' \overset{loc}{\rightrightarrows}$ na $\mathbb{R}$.

Tedy $g'(x) = \sum_{n=1}^{\infty} g_n'(x) = \sum_{n=1}^{\infty} \frac{-2x}{(n^2+x^2)^2}$, $x \in \mathbb{R}$.

Pak: Máme $f(x) = |x| \cdot g(x)$, tedy pro $x \neq 0$ $f'(x)$ existuje:

$f'(x) = \operatorname{sgn} x \cdot g(x) + |x| g'(x)$, $x \neq 0$.

Pro $x = 0$: $g(0) = \sum_{n=1}^{\infty} \frac{1}{n^2} \neq 0$, tedy $f(x) = |x| g(x)$ nemá derivaci v 0.

$\boxed{19}$ $f(x) = \sum_{n=1}^{\infty} \underbrace{\frac{\sin nx}{n^3}}_{f_n(x)}$, $x \in \mathbb{R}$ (a) $f \in C^1(\mathbb{R})$, (b) $f \in C^2((0,\pi))$

$f_n'(x) = \frac{\cos nx}{n^2}$

$f_n''(x) = \frac{-\sin nx}{n}$

$f(x)$ konverguje $\forall x \in \mathbb{R}$, $\boxed{1} \Rightarrow \sum f_n' \rightrightarrows$ na $\mathbb{R}$ $\Big\} \Rightarrow f \in C^1(\mathbb{R})$

$\boxed{11} \Longrightarrow \sum f_n'' \overset{loc}{\rightrightarrows}$ na $(0,\pi)$. $\Big\} \Rightarrow f \in C^2((0,\pi))$

$\boxed{20}$ $\zeta(x)=\sum_{n=1}^{\infty}\underbrace{\frac{1}{n^x}}_{f_n(x)}$, $x\in(1,\infty)$ $\qquad \zeta\in\mathcal{C}^{\infty}((1,\infty))$ ?

<u>Pro $a>1$</u>: $\quad \sup_{x\in[a,\infty)}|f_n(x)| = \frac{1}{n^a}$, $\quad \sum\frac{1}{n^a}\in\mathbb{R} \quad \Rightarrow \sum f_n \rightrightarrows [a,\infty)$

$\Rightarrow \sum f_n \overset{loc}{\rightrightarrows} (1,\infty) \quad \Rightarrow \zeta\in\mathcal{C}((1,\infty))$.

konstanta · lok. stejn. kon. řady

$f_n'(x)=\frac{-\log n}{n^x}$, ..., $f_n^{(k)}(x)=\frac{(-1)^k\log^k n}{n^x}$, $\quad \sum_{n=1}^{\infty} f_n^{(k)} \overset{loc}{\rightrightarrows}$ na $(1,\infty)$