Mocninné řady

Polynom $f(x) \overset{\text{def}}{=} \sum_{n=0} a_n x^n$, $x \in \mathbb{R}$

Mocninná řada $f(x) = \sum_{n=0}^{\infty} a_n (x - \underline{x_0})^n$, $x \in D_f$.

střed mocninné řady

poloměr konvergence.

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}$$

Z přednášky:

- $(x_0 - R, x_0 + R) \subset D_f \subset [x_0 - R, x_0 + R]$,
- $\sum_{n=0}^{\infty} a_n (x - x_0)^n \overset{loc}{\rightrightarrows}$ na $(x_0 - R, x_0 + R)$, včetně všech derivací

$\Rightarrow f \in C^\infty((x_0 - R, x_0 + R))$, $f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$, $f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$, ...

- Navíc: $f \in C(D_f)$, (tzn. i v krajních bodech, pokud tam jsou) – Abelova věta.

$\boxed{1}$ $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}$, $R = ?$, $x_0 = 0$, $a_n = \frac{1}{n^3}$.

$\boxed{\lim\limits_{n\to\infty} \sqrt[n]{n} = 1}$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n^3}} \overset{\uparrow}{=} 1 \Rightarrow R = \frac{1}{1} = 1.$$

$\Rightarrow$ řada konverguje pro $x \in (-1, 1)$, diverguje pro $x \notin [-1, 1]$,

v 1, -1 ?: $x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ konverguje,

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ konverguje. $\Rightarrow D_f = [-1, 1]$

Alternativní vzorec pro výpočet R: $R = \lim\limits_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$, pokud existuje.

$\boxed{2}$ $\sum_{n=1}^{\infty} \underbrace{\frac{(n!)^2}{(2n)!}}_{a_n} x^n$,

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{\frac{(n!)^2}{(2n)!}}{\frac{((n+1)!)^2}{(2(n+1))!}} = \lim_{n\to\infty} \frac{\frac{\cancel{(n!)^2}}{\cancel{(2n)!}}}{\frac{\cancel{(n!)^2}(n+1)^2}{\cancel{(2n)!}(2n+1)(2n+2)}} = 4 = R.$$

$\boxed{3}$ $\sum_{n=1}^{\infty} \frac{x^{n!}}{n!} = \sum_{k=1}^{\infty} a_k x^k$, $a_k = \begin{cases} \frac{1}{k}, & \text{pokud } k = n! \\ 0, & \text{jinak} \end{cases}$.

$$\limsup_{k\to\infty} \sqrt[k]{|a|_k} \le \limsup_{k\to\infty} \sqrt[k]{\frac{1}{k}} = 1 \Rightarrow R = \frac{1}{1} = 1$$

$$\ge \lim_{j\to\infty} \sqrt[j!]{\frac{1}{j!}} = 1$$

$\boxed{\text{Platí: } \limsup\limits_{n\to\infty} a_n \ge \lim\limits_{n_j\to\infty} n_j \quad \forall \text{ podposloupnost } n_j}$

$\boxed{4}$ $\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^p}}_{a_n} x^n$, $p \in \mathbb{R}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n\to\infty} e^{-p\frac{\log n}{n}\to 0} = e^0 = 1 \implies R = 1$$

$\boxed{5}$ $\sum_{n=1}^{\infty} \underbrace{\frac{3^n+(-2)^n}{n}}_{a_n} (x+1)^n$, $x_0 = -1$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{3^n} \sqrt[n]{\frac{1+\left(-\frac{2}{3}\right)^n \to 0}{n}} = \lim_{n\to\infty} 3 \frac{\sqrt[n]{1+\left|-\frac{2}{3}\right|^n} \to 1}{\sqrt[n]{n} \to 1} = 3 \implies R = \frac{1}{3}.$$

$\implies$ konverguje na $\left(-\frac{4}{3}, -\frac{2}{3}\right)$.

$\boxed{6}$ $\sum_{n=1}^{\infty} \underbrace{\left(1+\frac{1}{n}\right)^{n^2}}_{a_n} x^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{\cancel{\frac{n^2}{n}}\, n} = \lim_{n\to\infty} e^{\frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \to 1} = e^1 = e.$$

$R = \frac{1}{e}$.

$\boxed{7}$ $\sum_{n=1}^{\infty} \underbrace{\frac{1}{a^{\sqrt{n}}}}_{a_n} x^n$, $a > 0$:

$$\lim \sqrt[n]{a_n} = \lim \frac{1}{a^{\frac{\sqrt{n}}{n}}} = \lim \frac{1}{a^{\frac{1}{\sqrt{n}}} \to 0} = \frac{1}{a^0} = 1,$$

$\implies R = 1$

$\boxed{8}$ $\sum_{n=1}^{\infty} \underbrace{\frac{(3+(-1)^n)^n}{n}}_{a_n} x^n$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup \frac{3+(-1)^n}{\sqrt[n]{n}} \le \limsup \frac{3+1}{\sqrt[n]{n}} = 4$$

$$\ge \lim_{k\to\infty} \frac{3+(-1)^{2k} = 1}{\sqrt[2k]{2k} \to 1} = 4.$$

$\implies R = \frac{1}{4}$

$\boxed{9}$ $\sum_{n=1}^{\infty} \frac{x^n}{a^n+b^n}$, $a, b > 0$, $a_n = \frac{1}{a^n+b^n}$, BÚNO $a \ge b$.

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{a^n+b^n}} = \lim_{n\to\infty} \frac{1}{\underbrace{\sqrt[n]{a^n}}_{=a} \underbrace{\sqrt[n]{1+\left(\frac{b}{a}\right)^n}}_{\to 1}} = \frac{1}{a}. \implies R = a.$$

$$1 = \sqrt[n]{1} \le \sqrt[n]{1+\left(\frac{b}{a}\right)^n} \le \sqrt[n]{1+1} = \sqrt[n]{2} \xrightarrow{n\to\infty} 1$$

2 strážníci $\implies$

Pro obecné $a, b > 0$ tedy $R = \max\{a, b\}$.

$\boxed{10}$ $\sum_{n=1}^{\infty} \underbrace{\left(na^n + \frac{b^n}{n^2}\right)}_{a_n} x^n$, $0 < a < b$

$$b \xleftarrow{n\to\infty} \frac{b}{\sqrt[n]{n^2}} = \sqrt[n]{\frac{b^n}{n^2}} \le \sqrt[n]{|a_n|} = \sqrt[n]{na^n + \frac{b^n}{n^2}} \le \sqrt[n]{nb^n + \frac{b^n}{n^2}} =$$

$$= b \cdot \sqrt[n]{n + \frac{1}{n^2}} \le b \underbrace{\sqrt[n]{2}\sqrt[n]{n}}_{\xrightarrow{n\to\infty} 1}$$

2 strážníci $\implies \lim \sqrt[n]{|a_n|} = b$

$\implies R = \frac{1}{b}$.

$\boxed{11}$ $$\sum_{n=1}^{\infty} \frac{1}{2^n} x^{(n^2)} = \sum_{k=1}^{\infty} a_k x^k, \quad a_k = \begin{cases} \frac{1}{2^{\sqrt{k}}}, & k=n^2 \\ 0, & \text{jinak} \end{cases}$$

$$\limsup_{k\to\infty} \sqrt[k]{a_k} \le \limsup_{k\to\infty} \sqrt[k]{\frac{1}{2^{\sqrt{k}}}} = \lim_{k\to\infty} \frac{1}{2^{\frac{\sqrt{k}}{k}}} = 1 \qquad \frac{\sqrt{k}}{k} \to 0$$

$$\limsup_{k\to\infty} \sqrt[k]{a_k} \ge \lim_{n\to\infty} \sqrt[n^2]{a_{n^2}} = \lim_{n\to\infty} \sqrt[n^2]{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{2^{\frac{n}{n^2}}} = 1$$

$$\Rightarrow R = 1.$$

$\boxed{12}$ $\frac{1}{2-x}$ jako mocninnou řadu

Připomenutí: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad x \in (-1,1)$

(a) $\underline{x_0 = 0:}$ $$\frac{1}{2-x} = \frac{1}{2} \frac{1}{1-\frac{x}{2}} \underset{x\in(-2,2)}{=} \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n, \quad x \in (-2,2).$$

(b) $\underline{x_0 = 1:}$ $$\frac{1}{2-x} = \frac{1}{1-(x-1)} \underset{x\in(0,2)}{=} \sum_{n=0}^{\infty} (x-1)^n, \quad x \in (0,2)$$

Otázka: Máme zadanou funkci f. Lze zapsat f ve tvaru mocninné řady?

Poznámka: Nutné musí být $f \in C^\infty$. Ale $f \in C^\infty$ nestačí (viz $\boxed{21}$.)

Pro komplexní funkce platí: f' existuje $\Rightarrow$ lze to. (Viz komplexní analýza)

Jak vypadá mocninná řada f?

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \Rightarrow \quad f(x_0) = a_0$$

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} \quad \Rightarrow \quad f'(x_0) = 1 \cdot a_1$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)(x-x_0)^{n-k} \Rightarrow f^{(k)}(x_0) = k!\, a_k \Rightarrow \boxed{a_k = \frac{f^{(k)}(x_0)}{k!}}$$

$\Rightarrow$ Tedy řada musí být Taylorova řada f.

Otázku lze tedy přeformulovat i takto: $T_n^{f,a} \longrightarrow f$?

(13) $$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}, \qquad x \in \mathbb{R}$$

$\boxed{14}$ $$\frac{x^2+1}{x^2-1} = (x^2+1)\cdot\frac{-1}{1-x^2} \overset{x\in(-1,1)}{=} -(x^2+1)\sum_{n=0}^{\infty} x^{2n} = -\sum_{n=0}^{\infty} x^{2n+2} - \sum_{n=0}^{\infty} x^{2n} =$$
$$= -\sum_{k=1}^{\infty} x^{2k} - \sum_{n=0}^{\infty} x^{2n} = -1 - \sum_{n=1}^{\infty} 2x^{2n}$$

$$\frac{x^2+1}{x^2-1} = 1 + \frac{2}{x^2-1} \overset{x\in(-1,1)}{=} 1 - 2\sum_{n=0}^{\infty} x^{2n}.$$

$\boxed{15}$ $$\frac{x}{\sqrt{1-2x}} = x(1-2x)^{-\frac{1}{2}} \underset{x\in(-\frac{1}{2},\frac{1}{2})}{=} x\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-2x)^n = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-2)^n x^{n+1}.$$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n, \quad x\in(-1,1).$$

(16)
$$\frac{1}{(1+x^2)^2} = (1+x^2)^{-2} \overset{x\in(-1,1)}{=} \sum_{n=0}^{\infty}\binom{-2}{n}x^{2n}$$

$$\frac{1}{(1+x^2)^2} = \frac{1}{1+x^2}\cdot\frac{1}{1+x^2} \overset{x\in(-1,1)}{=} \sum_{n=0}^{\infty}(-1)^n x^{2n} \cdot \sum_{m=0}^{\infty}(-1)^m x^{2m} = (*)$$

[Věta (Abel) $\sum_{n=\cancel{1}\,0}^{\infty} a_n \cdot \sum_{m=\cancel{1}\,0}^{\infty} b_m = \sum_{k=\cancel{1}\,0}^{\infty}\left(\sum_{i=\cancel{1}\,0}^{k} a_{k\cancel{+1}-i}\, b_i\right)$, pokud všechny řady konvergují

Mertens ⇒ stačí $\sum a_n$ k, $\sum b_n$ AK. ⇒ pro mocninné řady lze, na vnitřku intervalu konvergence]

$$(*) = \sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)^{k-i}x^{2(k-i)}(-1)^i x^{2i} = \underbrace{\sum_{k=0}^{\infty}\Bigl(\sum_{i=0}^{k}\underbrace{(-1)^{k-i}(-1)^i}_{(-1)^k}}_{(k+1)(-1)^k}\Bigr)x^{2k} = \sum_{k=0}^{\infty}(k+1)(-1)^k x^{2k},$$
$$x\in(-1,1).$$

$\boxed{17}$ $$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2x)^{2n} = \frac{1}{2} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}2^{2n-1}}{(2n)!}x^{2n}, \quad x\in\mathbb{R}.$$

$\boxed{18}$ $\log x,\quad x_0 = 1$, $\qquad$ [$\log(x+1),\ x_0 = 0$]

$$(\log x)' \underset{x\in(0,\infty)}{=} \frac{1}{x} = \frac{1}{1-(1-x)} \underset{x\in(0,2)}{=} \sum_{n=0}^{\infty}(1-x)^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n. \qquad \Big/ \int dx$$

$$\log x = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(x-1)^{n+1} + C, \quad x\in(0,2).$$

$x := 1 \Rightarrow 0 = \log 1 = 0 + C$
$\Rightarrow C = 0$

$$\Rightarrow \log x = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(x-1)^k, \quad x\in(0,2).$$ Co pro $x = 2$?

Pro $x=2$: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \in \mathbb{R}$ (Leibniz).

$$\log 2 = \lim_{x\to 2^-} \log(x) \overset{\text{viz výše.}}{=} \lim_{x\to 2^-} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1) \overset{\text{Abelova věta}}{=} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

Tedy $\displaystyle \log x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k, \quad x \in (0,2]$.

$\boxed{19}$
$$\frac{x}{1+x-2x^2} = x \cdot \frac{1}{(1-x)} \cdot \frac{1}{(1+2x)} \overset{x\in(-\frac{1}{2},\frac{1}{2})}{=} x \sum_{n=0}^{\infty} x^n \cdot \sum_{m=0}^{\infty} (-2x)^m = x \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} (-2)^i\right) x^k =$$
$$= x \sum_{k=0}^{\infty} \frac{1-(-2)^{k+1}}{1-(-2)} x^k = \sum_{k=0}^{\infty} \frac{1}{3}\left(1-(-2)^{k+1}\right) x^{k+1}, \quad x \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right).$$

$\boxed{20}$
$$\log(1+x+x^2+x^3) \underset{x+x^2+x^3\in(-1,1)}{=} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x+x^2+x^3)^n \dots$$

$$\log(1+x+x^2+x^3) = \log\left[(1+x)(1+x^2)\right] = \log(1+x) + \log(1+x^2) \underset{x\in(-1,1]}{=} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^{2n}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x^n + x^{2n})$$

$\boxed{21}$
$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

Ukážeme, že $f \in \mathcal{C}^\infty(\mathbb{R})$, avšak $\displaystyle f(x) \neq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ pro $x \neq 0$.

- f je spojitá i v 0 ✓
- $f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}$, $\quad \left.\begin{array}{l} \lim_{x\to 0} f'(x) = 0 \\ f \text{ spojitá v } 0 \end{array}\right\} \Rightarrow f \in \mathcal{C}^1(\mathbb{R})$, $f'(0)=0$
- $f''(x) = \left(-\frac{6}{x^4} + \frac{2}{x^3}\right) e^{-\frac{1}{x^2}}$, $\quad \left.\begin{array}{l} \lim_{x\to 0} f''(x) = 0 \\ f' \text{ spojitá} \end{array}\right\} \Rightarrow f \in \mathcal{C}^2(\mathbb{R})$, $f''(0)=0$

$\vdots$

- $f^{(k)}(x) = \left(\frac{c_k}{x^{k+2}} + \dots + \frac{c_1}{x^3}\right) e^{-\frac{1}{x^2}}$, $\quad \left.\begin{array}{l} \lim_{x\to 0} f^{(k)}(x) = 0 \\ f^{(k-1)} \text{ spojitá} \end{array}\right\} \Rightarrow f \in \mathcal{C}^k(\mathbb{R})$, $f^{(k)}(0)=0$

Tedy $f \in \mathcal{C}^\infty(\mathbb{R})$

$\Rightarrow$ Taylorova řada je $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0, \quad x \in \mathbb{R}$

Ale $f(x) \neq 0, \quad x \in \mathbb{R} \setminus \{0\}$.

Sčítání řad

$\boxed{1}$ $$\sum_{n=1}^{\infty} n x^n = ?$$

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' \overset{x\in(-1,1)}{=} \left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=1}^{\infty} n x^{n-1} \implies \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}, \quad x \in (-1,1)$$

$\boxed{2}$ $$\sum_{n=1}^{\infty} \frac{x^n}{n} = ?$$

$$-\log(1-x) \overset{c}{=} \int \frac{1}{1-x}\,dx \overset{x\in(-1,1)}{=} \int \sum_{n=0}^{\infty} x^n\,dx \overset{c}{=} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$C = ?$: $x := 0 \implies 0 = -\log 1 = 0 + C \rightsquigarrow C = 0$

$$\implies \sum_{k=1}^{\infty} \frac{x^k}{k} = -\log(1-x), \quad x \in (-1,1)$$

Pro $x=-1$ konverguje, Abel $\implies x \in [-1,1)$.