The Hazards and Challenges of Low-Precision Computation

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FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

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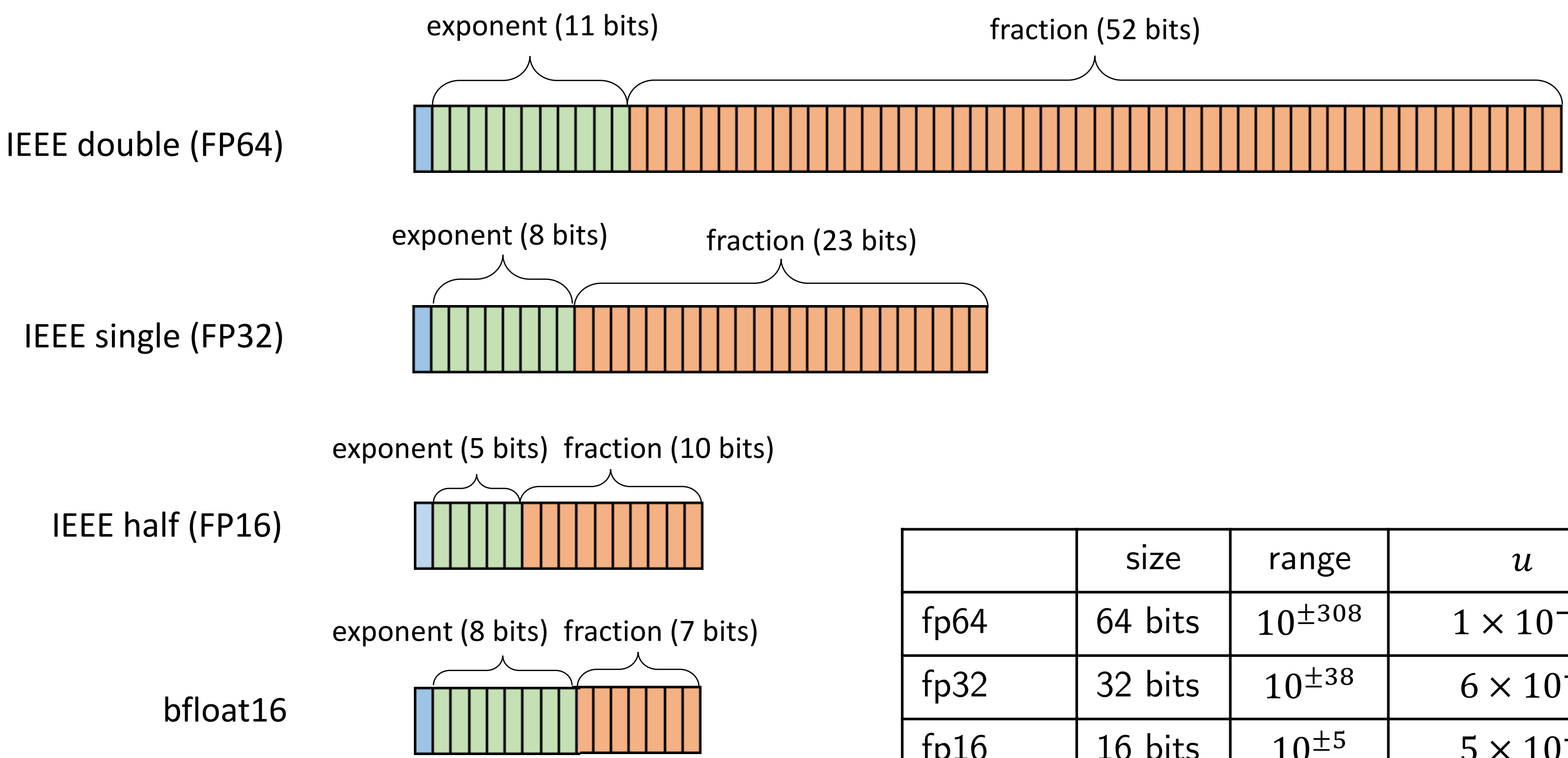
Floating Point Formats

$$(-1)^{\mathrm{sign}} \times 2^{(\mathrm{exponent}-\mathrm{offset})} \times 1.\mathrm{fraction}$$



	size	range	u
fp64	64 bits	$10^{\pm 308}$	1×10^{-16}
fp32	32 bits	$10^{\pm 38}$	6×10^{-8}
fp16	16 bits	$10^{\pm 5}$	5×10^{-4}
bfloat16	16 bits	$10^{\pm 38}$	4×10^{-3}

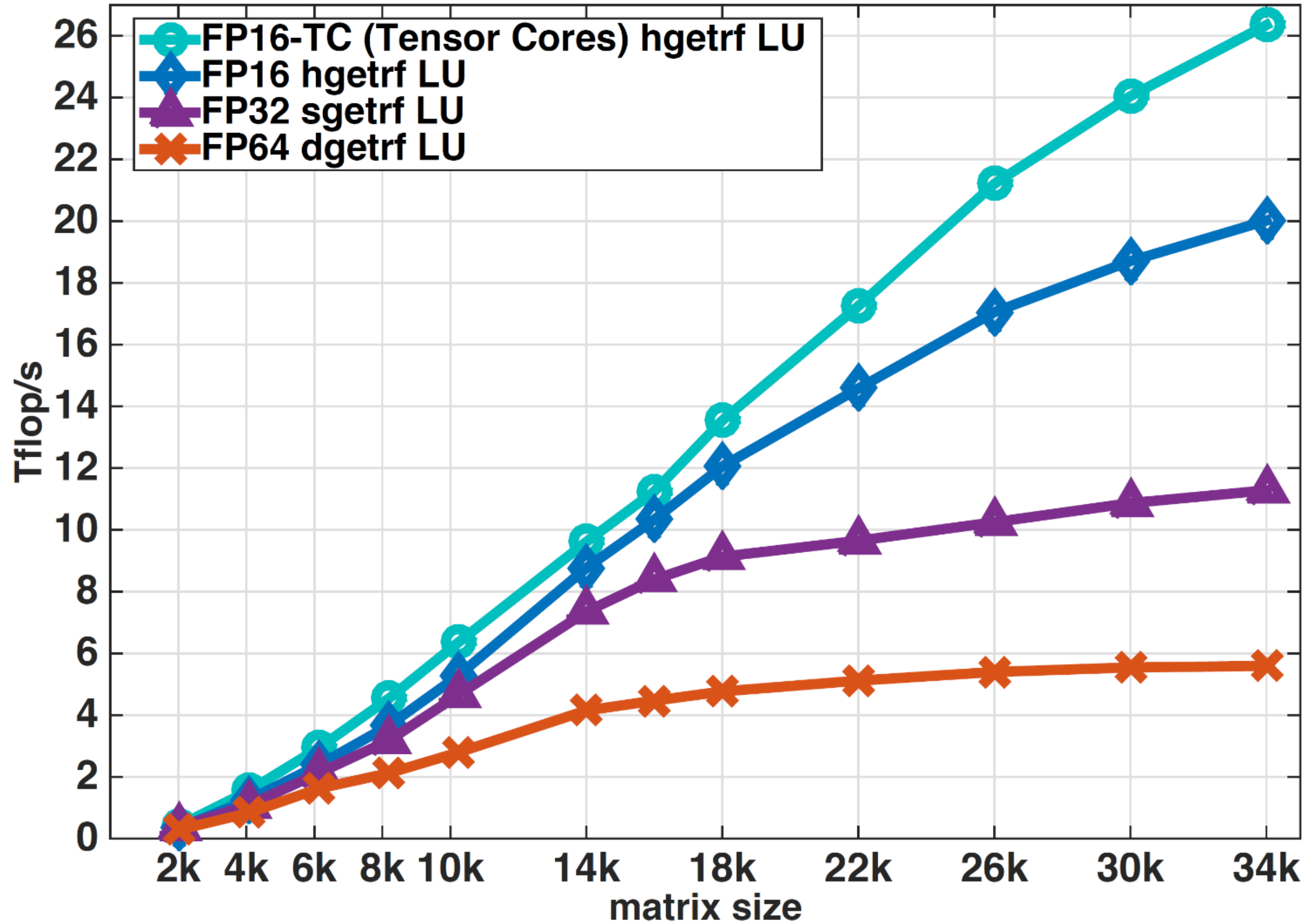
Hardware Support for Multiprecision Computation

Use of low precision in machine learning has driven emergence of low-precision capabilities in hardware:

- AMD Radeon Instinct MI25 GPU, 2017:
  - single: 12.3 TFLOPS, half: 24.6 TFLOPS
- NVIDIA Tesla P100, 2016: native ISA support for 16-bit FP arithmetic
- NVIDIA Tesla V100, 2017: tensor cores for half precision;

  4x4 matrix multiply in one clock cycle
  - double: 7 TFLOPS, half+tensor: 112 TFLOPS (**16x!**)
- NVIDIA A100, 2020: tensor cores with multiple supported precisions: FP16, FP64, Binary, INT4, INT8, bfloat16
- Intel AI processors (Nervana, Xeon)
- Google's Tensor processing unit (TPU): as low as 8-bit arithmetic, bfloat16
- Future exascale supercomputers: (~2021) Expected extensive support for reduced-precision arithmetic (32/16/8-bit)

Performance of LU factorization on an NVIDIA V100 GPU



[Haidar, Tomov, Dongarra, Higham, 2018]

Mixed Precision Capabilities on Supercomputers

From TOP500:

June 2021

	Accelerator/CP Family	Count	System Share (%)	Rmax (GFlops)	Rpeak (GFlops)	Cores
1	NVIDIA Volta	97	19.4	626,503,420	1,049,977,600	11,875,056
2	NVIDIA Ampere	26	5.2	351,252,600	505,841,268	3,435,116
3	NVIDIA Pascal	9	1.8	57,876,640	85,807,525	1,141,300

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1	NVIDIA Pascal	61	12.2	106,025,166	179,951,012	2,738,356
3	NVIDIA Volta	12	2.4	224,559,400	360,593,742	4,488,720

HPL-AI Benchmark

- Highlights confluence of HPC+AI workloads
  - Like HPL, solves dense Ax=b, results still to double precision accuracy
  - Achieves this via mixed-precision iterative refinement
    - may be implemented in a way that takes advantage of the current and upcoming devices for accelerating AI workloads

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- HPL-AI Results (June 2021):
  1. Fugaku: 2 EXAFLOP/s (vs. 442 PETAFLOP/s on HPL; 4.5x)
  2. Summit: 1.15 EXAFLOP/s (vs. 149 PETAFLOP/s on HPL; 7.7x)

- More information: https://icl.bitbucket.io/hpl-ai/
- Reference implementation: https://bitbucket.org/icl/hpl-ai/src/

Mixed precision in NLA

- BLAS: cuBLAS, MAGMA, [Agullo et al. 2009], [Abdelfattah et al., 2019], [Haidar et al., 2018]
- Iterative refinement:
  - Long history: [Wilkinson, 1963], [Moler, 1967], [Stewart, 1973], ...
  - More recently: [Langou et al., 2006], [C., Higham, 2017], [C., Higham, 2018], [C., Higham, Pranesh, 2020], [Amestoy et al., 2021]
- Matrix factorizations: [Haidar et al., 2017], [Haidar et al., 2018], [Haidar et al., 2020], [Abdelfattah et al., 2020]
- Eigenvalue problems: [Dongarra, 1982], [Dongarra, 1983], [Tisseur, 2001], [Davies et al., 2001], [Petschow et al., 2014], [Alvermann et al., 2019]
- Sparse direct solvers: [Buttari et al., 2008]
- Orthogonalization: [Yamazaki et al., 2015]
- Multigrid: [Tamstorf et al., 2020], [Richter et al., 2014], [Sumiyoshi et al., 2014], [Ljungkvist, Kronbichler, 2017, 2019]
- (Preconditioned) Krylov subspace methods: [Emans, van der Meer, 2012], [Yamagishi, Matsumura, 2016], [C., Gergelits, Yamazaki, 2021], [Clark, 2019], [Anzt et al., 2019], [Clark et al., 2010], [Gratton et al., 2020], [Arioli, Duff, 2009], [Hogg, Scott, 2010]

For survey and references, see [Abdelfattah et al., IJHPC, 2021]

Challenges of low precision

- Do error bounds still apply?
  - Error bound with constant nu provides no information if $nu > 1$
  - One solution: probabilistic approach [Higham, Mary, 2019], [Higham, Mary, 2020]

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- Larger unit roundoff
  - Lose something small when storing: $fl(x) = x(1+\delta), \ \ |\delta| \leq u$
  - Lose something small when computing: $fl(x \text{ op } y) = (x \text{ op } y)(1+\delta), \ \ |\delta| \leq u$

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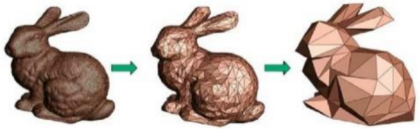
Does it matter?

Inexact computations

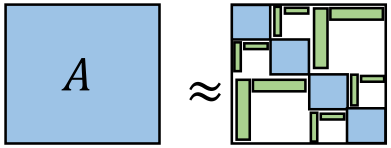
- In real computations we have many sources of inexactness
    - Imperfect data, measurement error
    - Modeling error, discretization error
    - Intentional approximation to improve performance
        - Reduced models, Low-rank representations, sparsification, randomization

Model Reduction

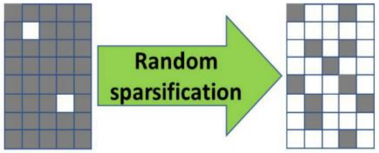


[Schilders, van der Vorst, Rommes, 2008]

Low-rank (hierarchical) approximation



Sparsification, Randomized algorithms



[Sinha, 2018]

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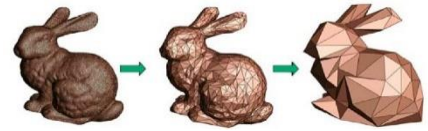

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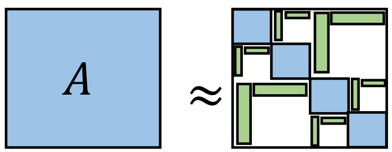

Sparsification, Randomized algorithms

[Sinha, 2018]

- Given that we are already working with so much inexactness, does it matter if we use lower precision?
  - Analysis of accuracy in techniques that use intentional approximation ***almost always*** assume that roundoff error is small enough to be ignored
  - Is this true? Is it true even if we use low precision?

Example: Randomized Algorithms

- Given $m \times n$ A, want truncated SVD with parameter k

$$A \approx \hat{U} \hat{\Sigma} \hat{V}^T$$

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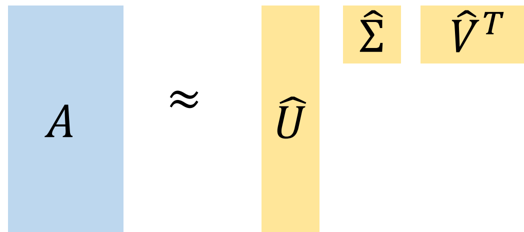
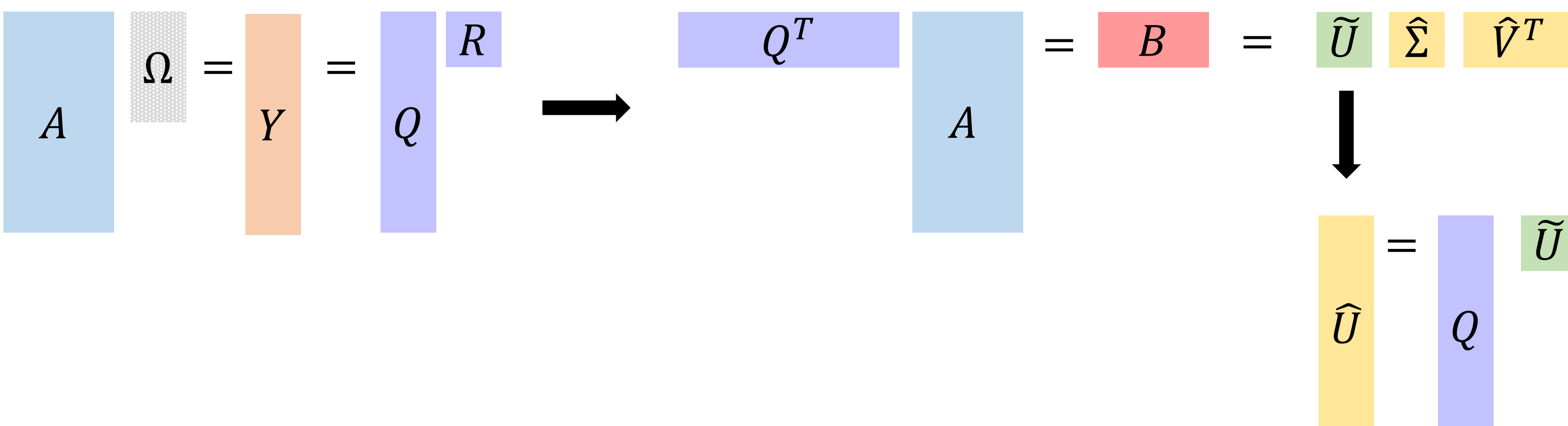


- Randomized SVD:



Assuming exact arithmetic:
If Q satisfies $\|A - QQ^T A\| \leq \varepsilon$, then $\left\|A - \hat{U}\hat{\Sigma}\hat{V}^T\right\| \leq \varepsilon$

What happens in finite precision?

Let's try different types of randsvd matrices from the MATLAB gallery:

```
A = gallery('randsvd',[100,40],1e6,mode); k=15;
```

$[U, S, V] \quad = \mathrm{svd}(A)$: non-randomized SVD, exact arithmetic

$[\hat{U}, \hat{S}, \hat{V}] \quad = \mathrm{rsvd}(A)$: randomized SVD, exact arithmetic

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Mode 3: Geometrically distributed singular values

$\|A - USV^T\|_2$ = 4.92e-03

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Use of low precision leads to an order magnitude loss of accuracy! Roundoff error can't be ignored!

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Error bound no longer holds!

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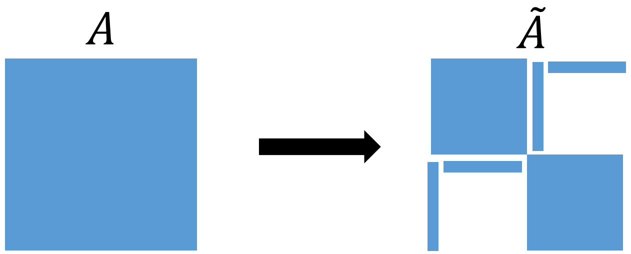
Example: Low-Rank Approximation

- Block low-rank approximation and hierarchical matrix representations arise in a variety of applications



- Work on mixed and low precision in block low-rank computations

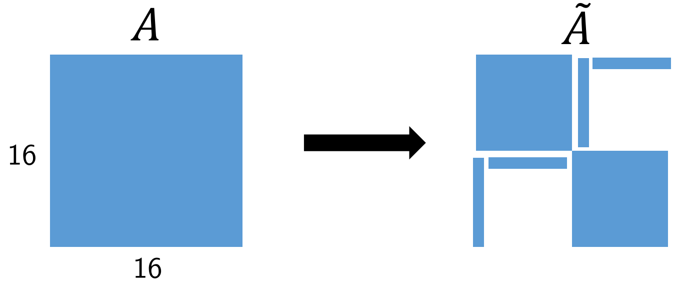
- [Higham, Mary, 2019]: block low-rank LU factorization preconditioner that exploits numerically low-rank structure of the error for LU computed in low precision
- [Higham, Mary, 2019]: Interplay of roundoff error and approximation error in solving block low-rank linear systems using LU
- [Buttari, et al., 2020]: block low-rank single precision coarse grid solves in multigrid
- [Buttari et al., 2021]: Mixed precision low rank approximation and application to block low-rank LU factorization

Example: Low-Rank Approximation

Inverse multiquadratic kernel:

$$A(i,j) = \frac{1}{\sqrt{1 + 0.1\|x - y\|^2}}, \qquad x, y \in \mathbb{R}^2$$

A is SPD. Low-rank approximation of A should also be SPD!



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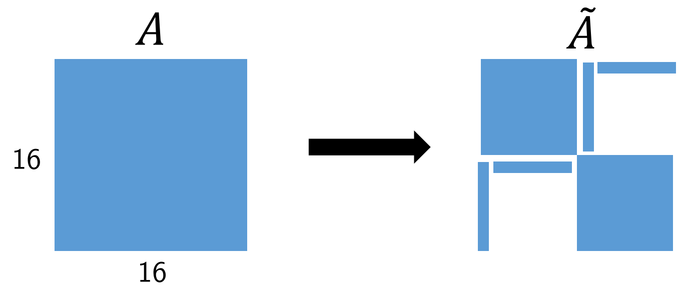
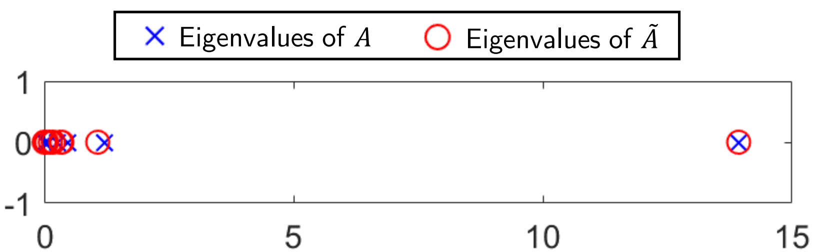


Exact arithmetic SVD:



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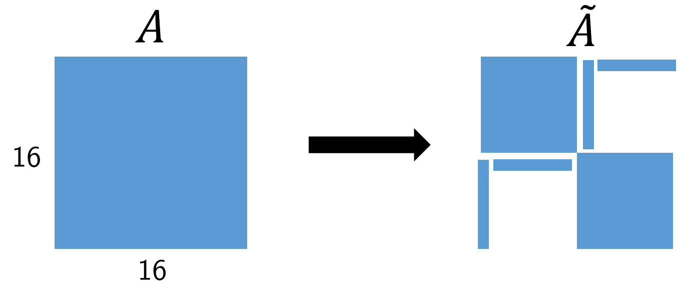

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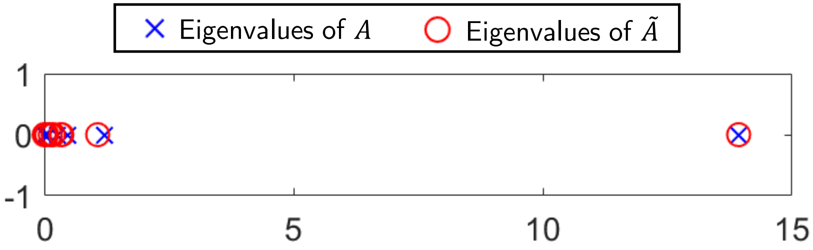

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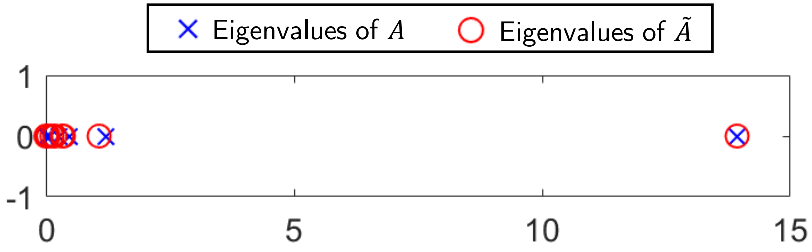

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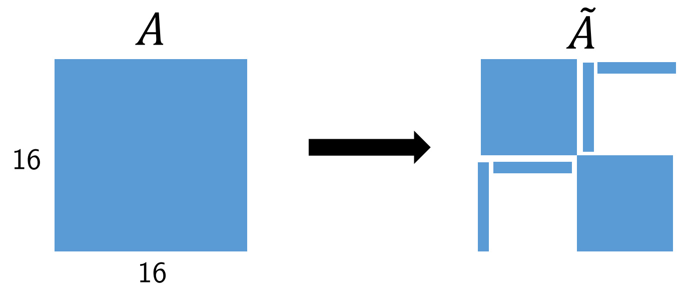

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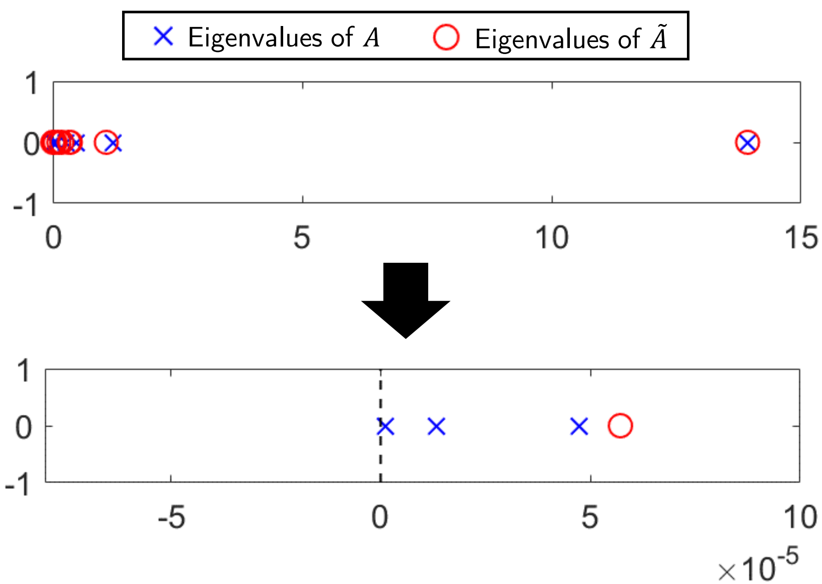

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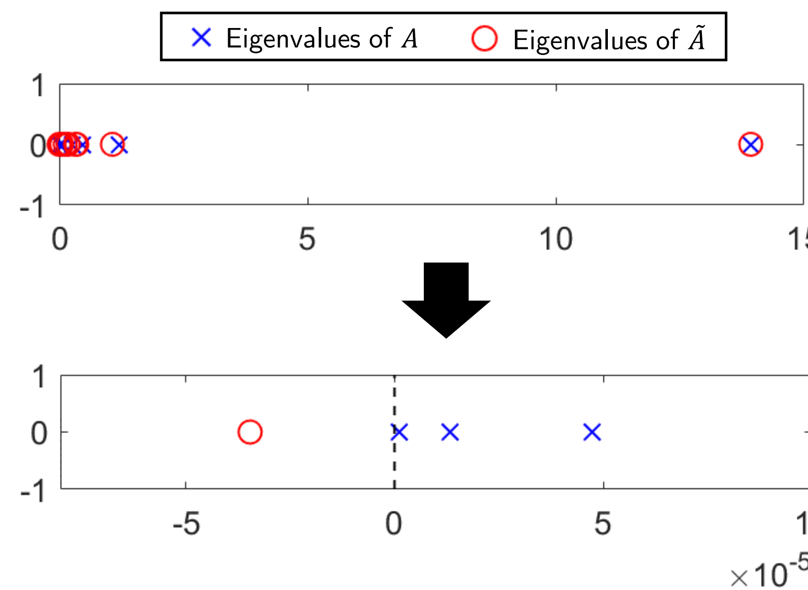

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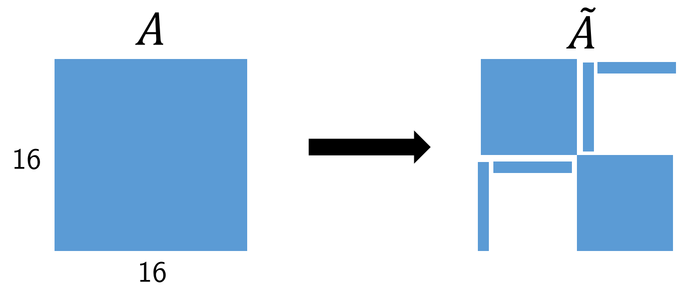

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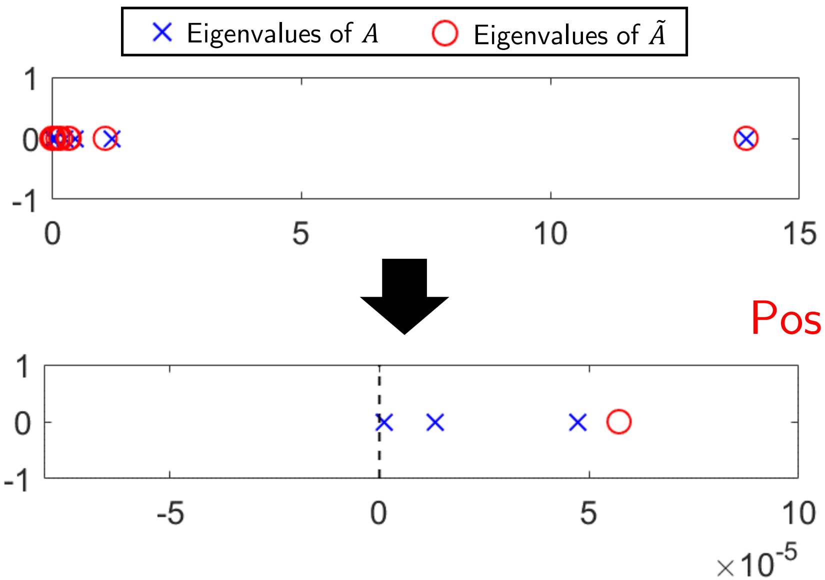
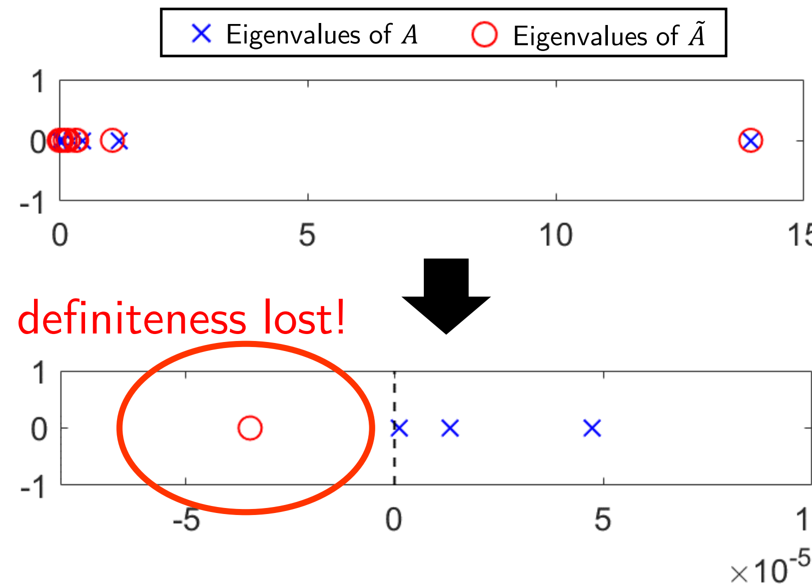


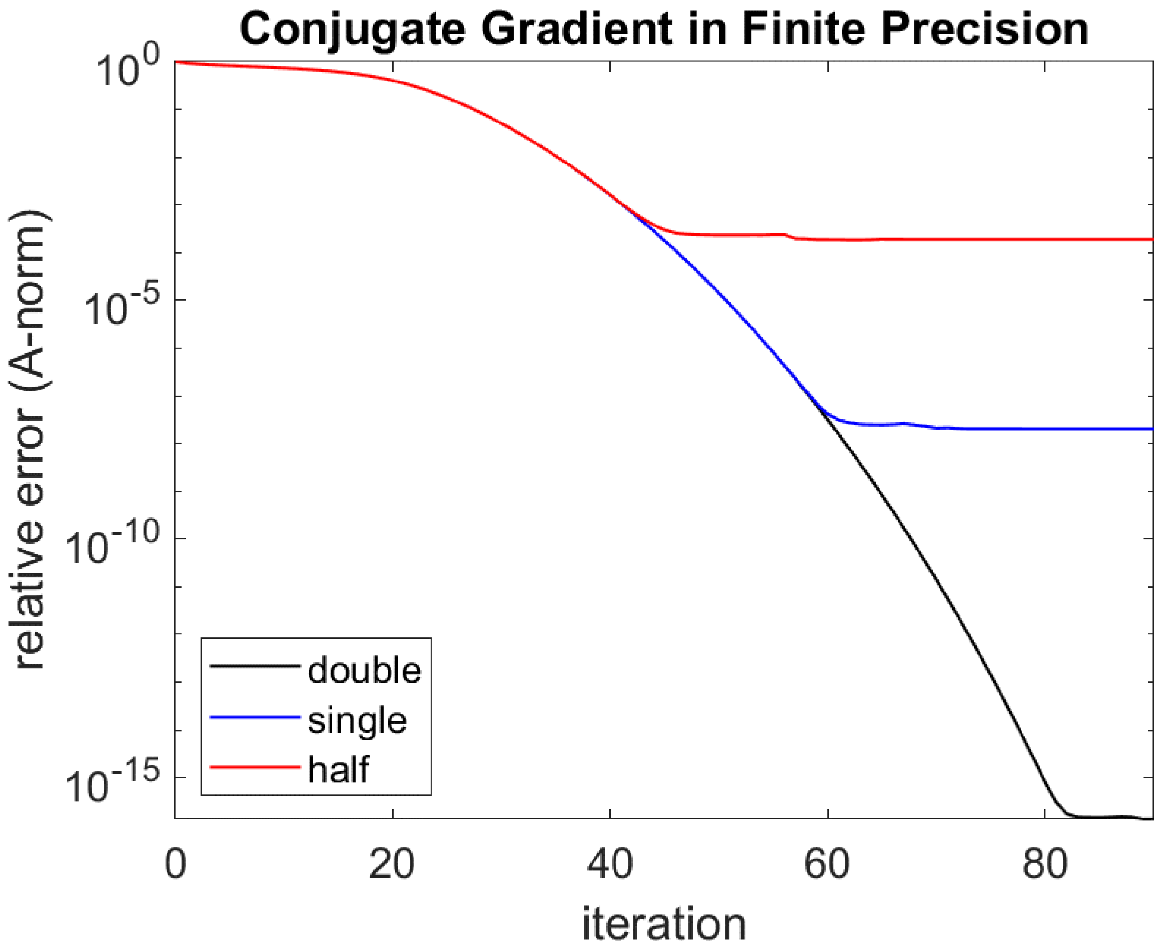
Half precision SVD:

Positive definiteness lost!



Example: Iterative Methods

```
A = diag(linspace(.001,1,100));
[V,~] = eig(A);
b = V'*ones(n,1);
```



Example: Iterative Methods

$n = 100, \lambda_1 = 10^{-3}, \lambda_n = 1$

$\lambda_i = \lambda_1 + \left(\frac{i-1}{n-1}\right)(\lambda_n - \lambda_1)(0.65)^{n-i}, \quad i = 2, \dots, n-1$

```
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b = V'*ones(n,1);
```

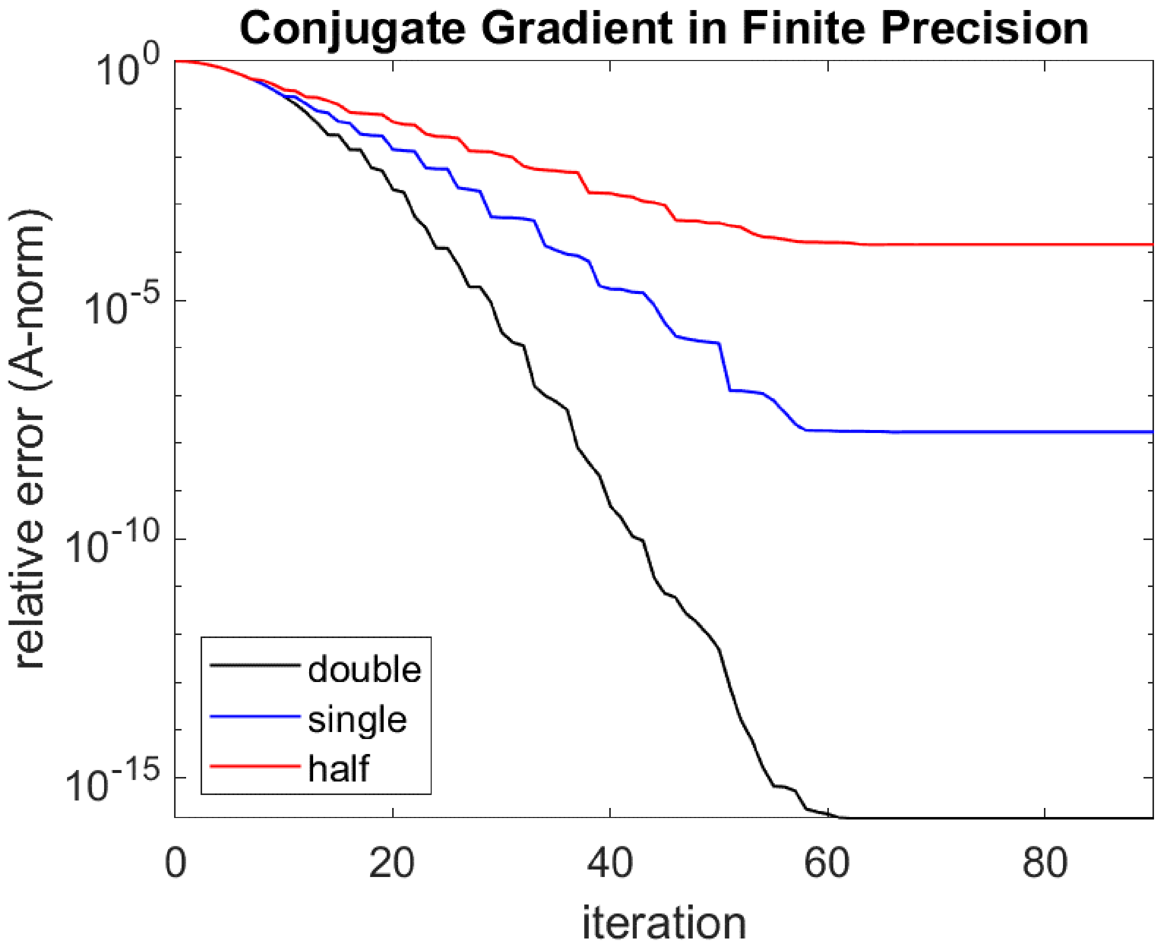

Takeaway

- Low precision can have massive performance benefits but must be used with caution!

- Many opportunities for using mixed and low precision computation in scientific applications

- Need to develop a theoretical understanding of how mixed precision algorithms behave; need to revisit analyses of algorithms and techniques that ignore finite precision

Thank you!

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