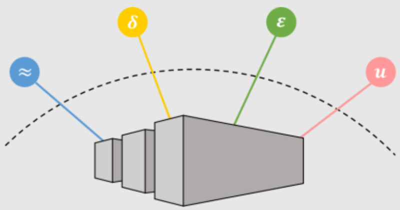

Balancing Inexactness in Matrix Computations

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Charles University

ILAS 2023
June 15, 2023

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

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European Union

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The Exascale Era

We have now entered the "Exascale Era"

- 10^{18} floating point operations per second

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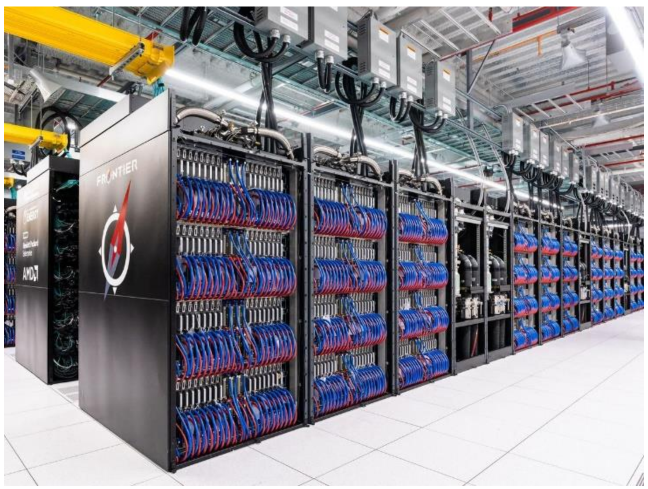

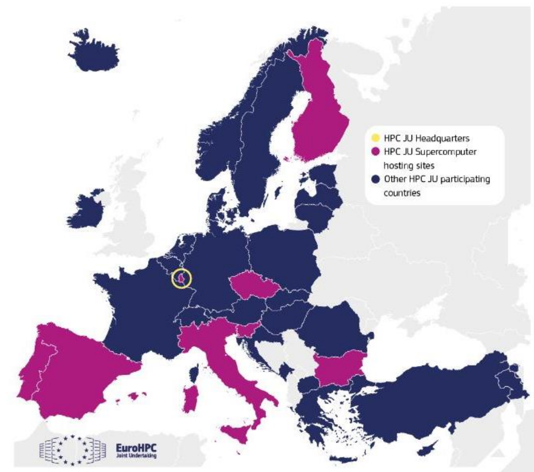

https://eurohpc-ju.europa.eu/pictures

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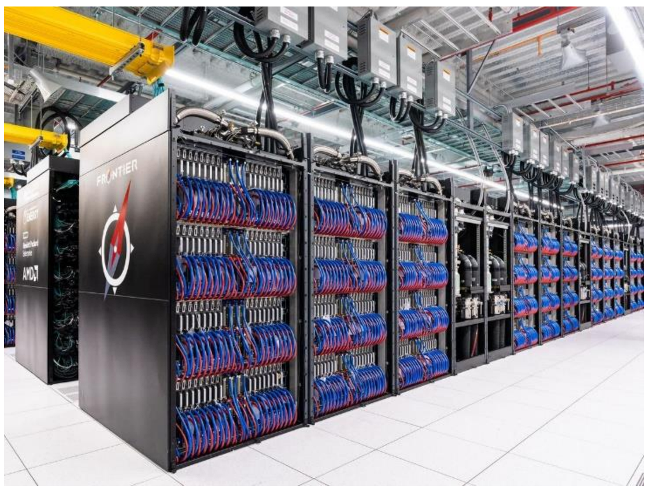

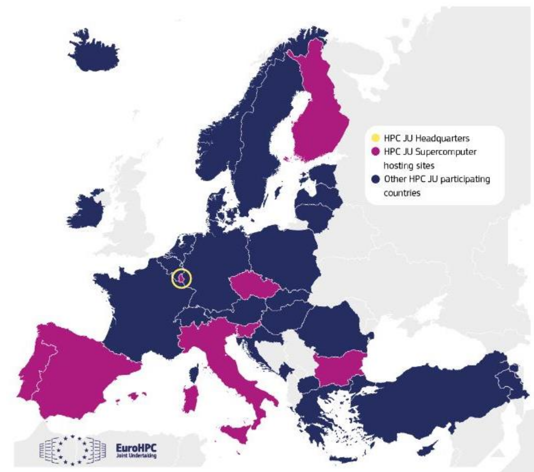

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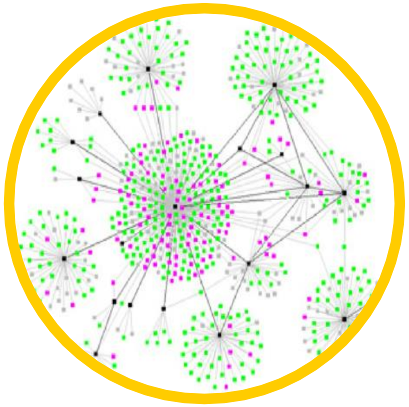
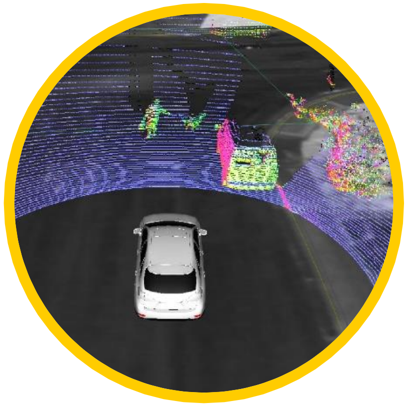
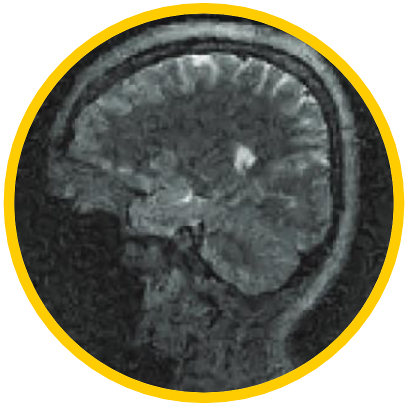
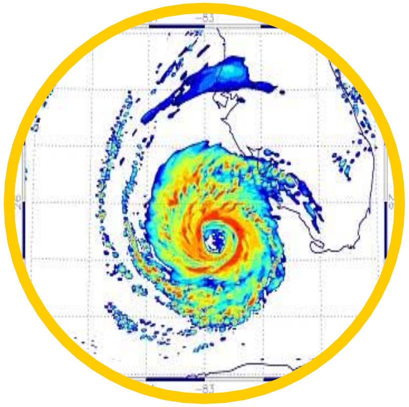
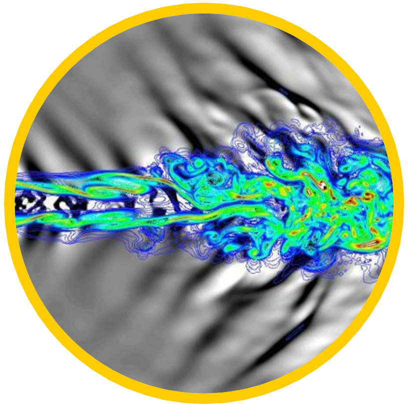
Significant opportunity ... Significant challenges

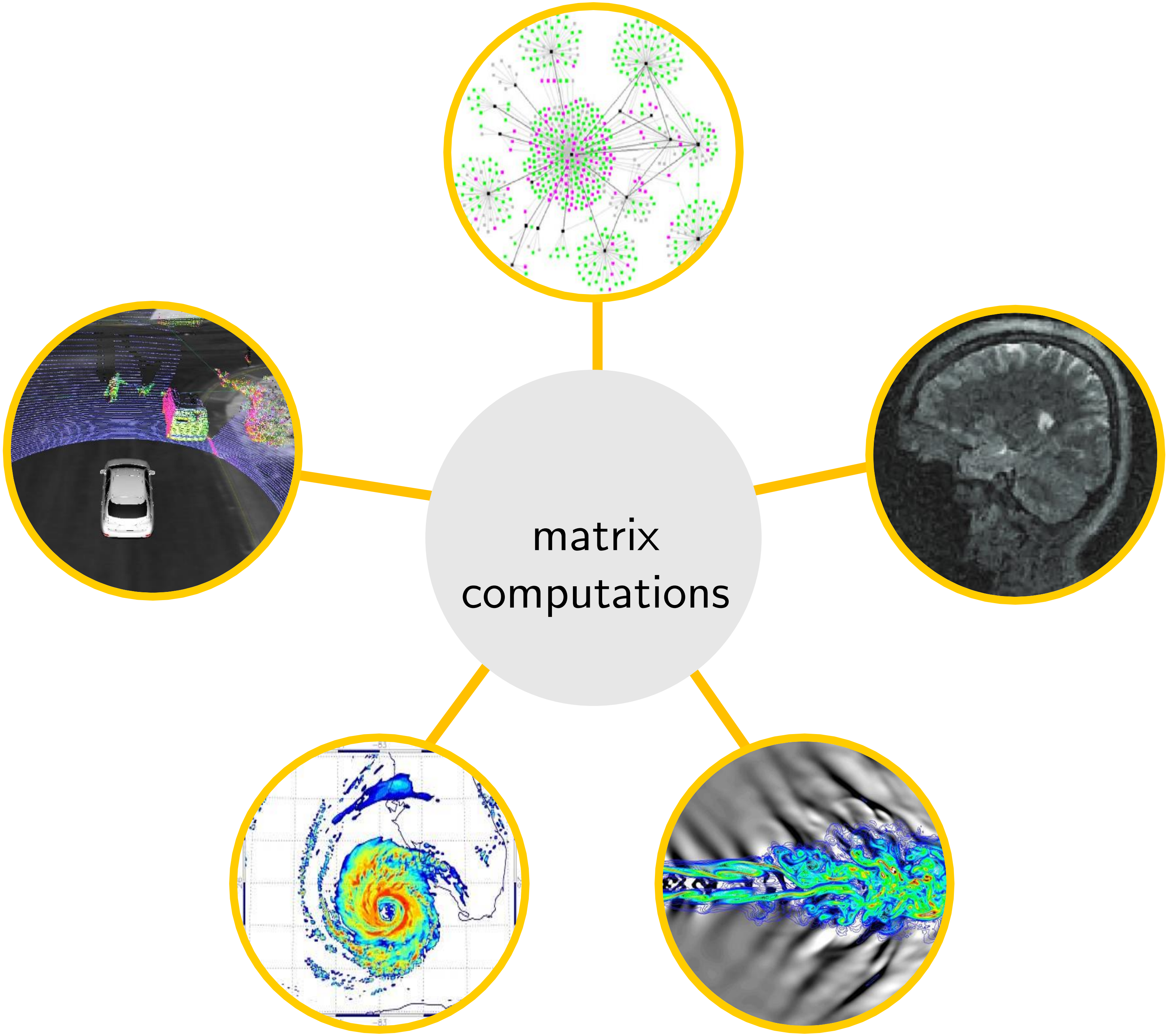








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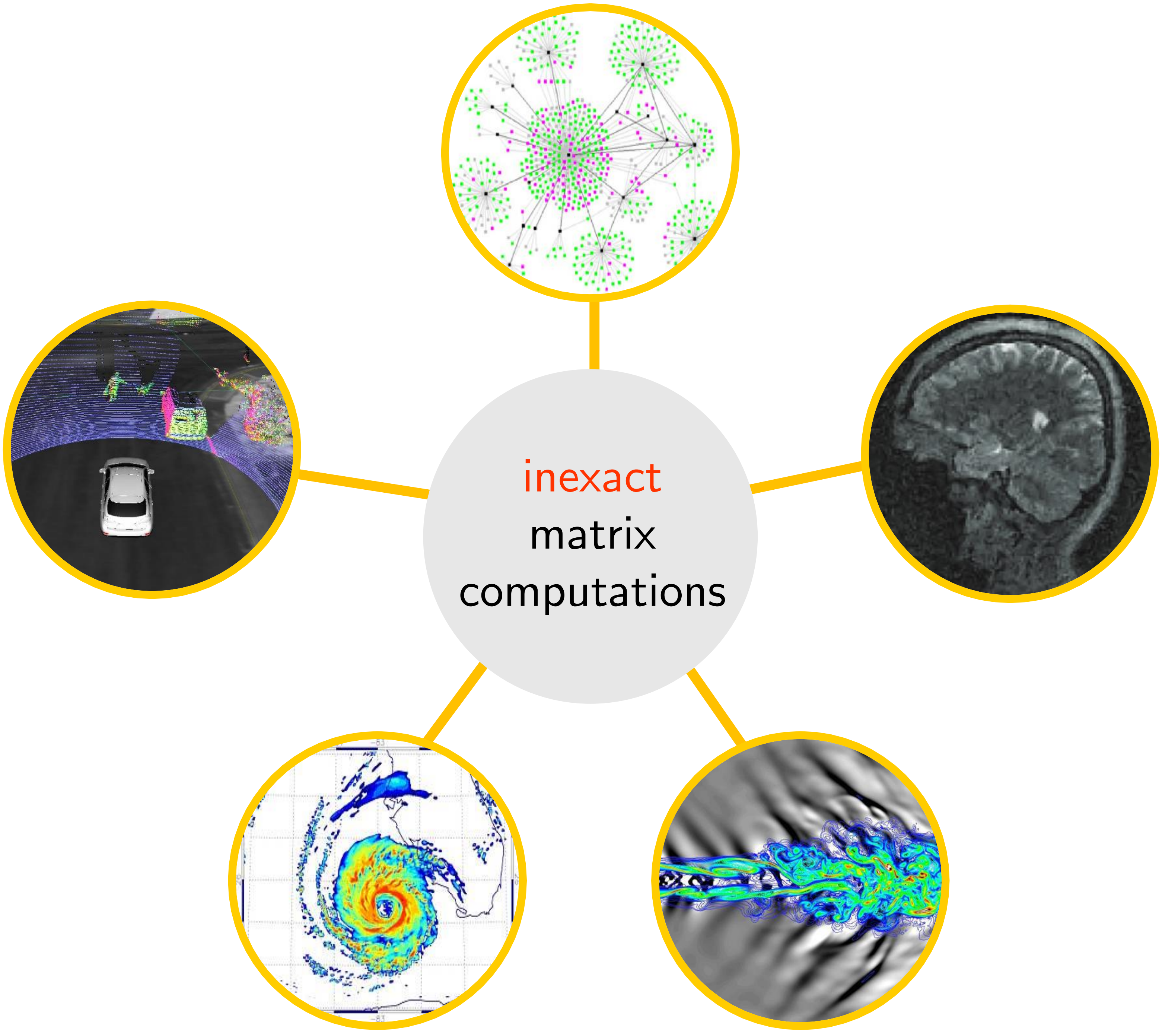
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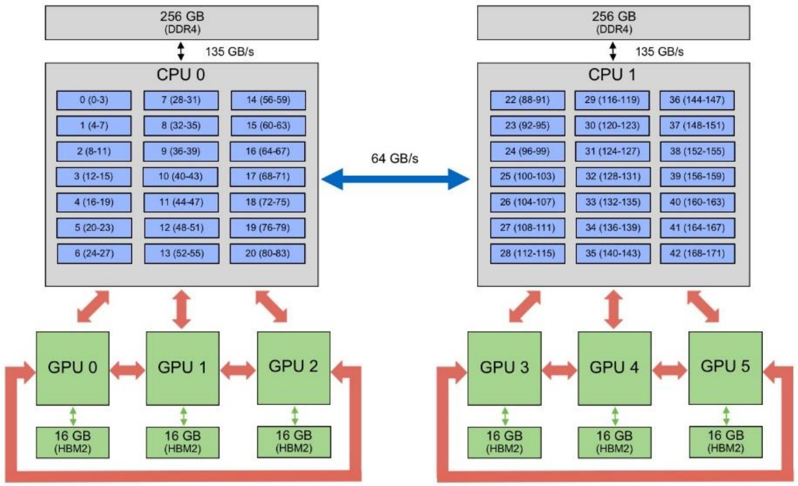
Exascale Hardware



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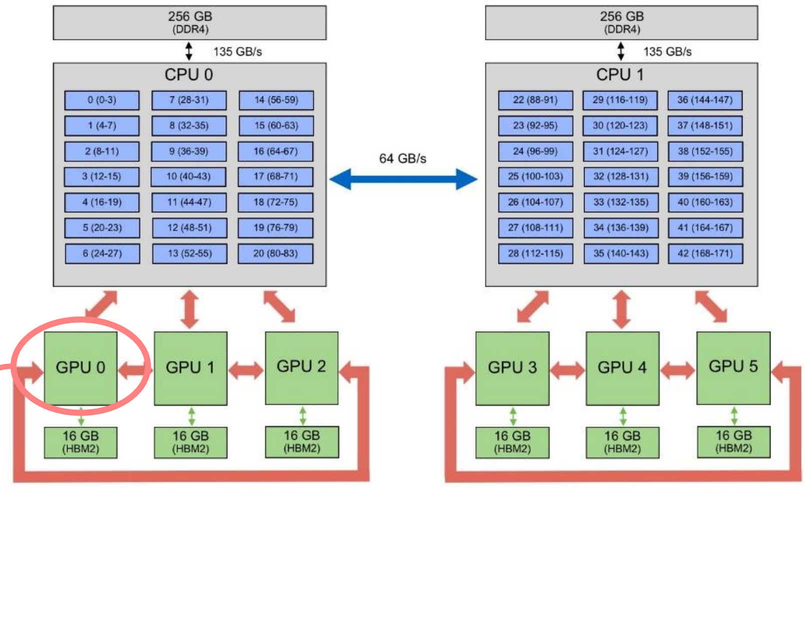
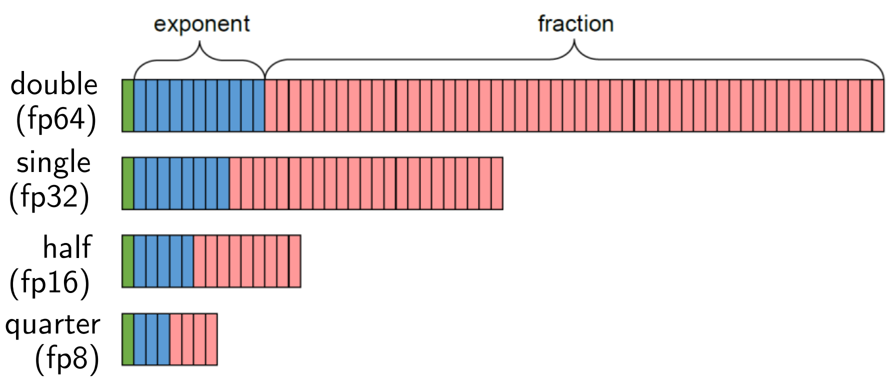
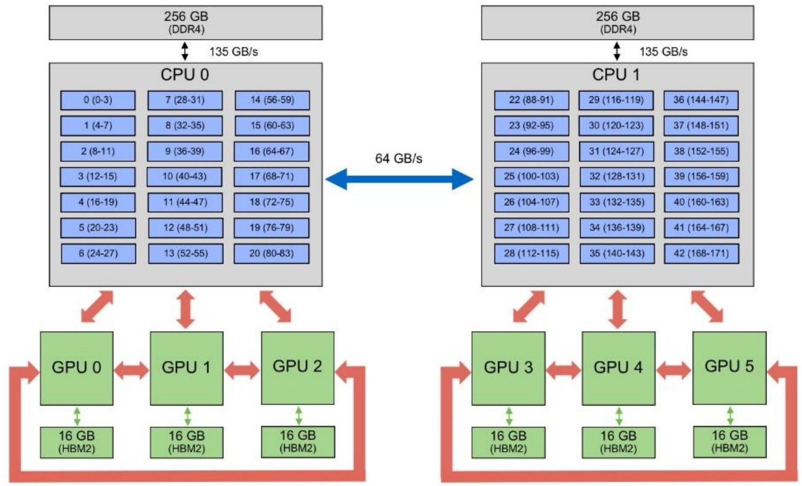


$$(-1)^{\text{sign}} \times 2^{(\text{exponent}-\text{offset})} \times 1.\text{fraction}$$



	size (bits)	range	u	perf. (NVIDIA H100)
fp64	64	$10^{\pm 308}$	1×10^{-16}	60 Tflops/s
fp32	32	$10^{\pm 38}$	6×10^{-8}	1 Pflop/s
fp16	16	$10^{\pm 5}$	5×10^{-4}	2 Pflops/s
bfloat16	16	$10^{\pm 38}$	4×10^{-3}	
fp8-e5m2	8	$10^{\pm 5}$	1×10^{-1}	4 Pflops/s
fp8-e4m3	8	$10^{\pm 2}$	6×10^{-2}	

Exascale Hardware



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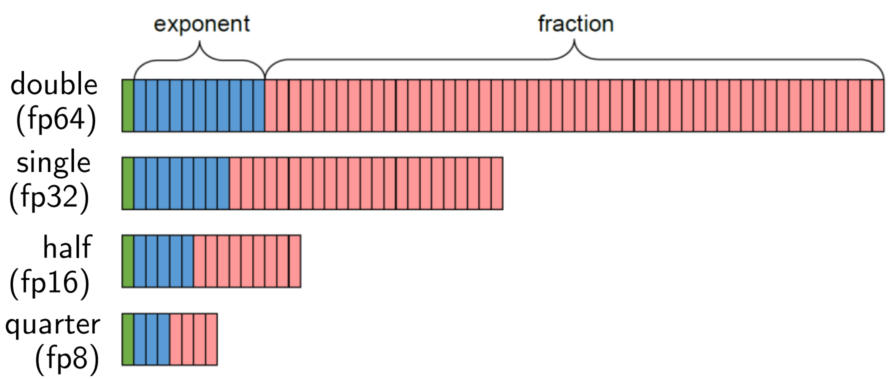

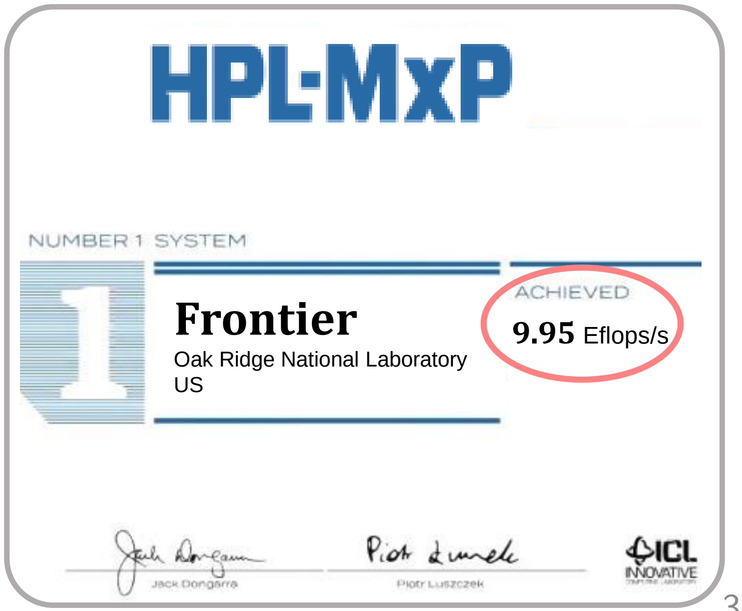

Mixed precision in NLA

- BLAS: cuBLAS, MAGMA, [Agullo et al. 2009], [Abdelfattah et al., 2019], [Haidar et al., 2018]
- Iterative refinement:
  - Long history: [Wilkinson, 1963], [Moler, 1967], [Stewart, 1973], ...
  - More recently: [Langou et al., 2006], [C., Higham, 2017], [C., Higham, 2018], [C., Higham, Pranesh, 2020], [Amestoy et al., 2021]
- Matrix factorizations: [Haidar et al., 2017], [Haidar et al., 2018], [Haidar et al., 2020], [Abdelfattah et al., 2020]
- Eigenvalue problems: [Dongarra, 1982], [Dongarra, 1983], [Tisseur, 2001], [Davies et al., 2001], [Petschow et al., 2014], [Alvermann et al., 2019]
- Sparse direct solvers: [Buttari et al., 2008]
- Orthogonalization: [Yamazaki et al., 2015]
- Multigrid: [Tamstorf et al., 2020], [Richter et al., 2014], [Sumiyoshi et al., 2014], [Ljungkvist, Kronbichler, 2017, 2019]
- (Preconditioned) Krylov subspace methods: [Emans, van der Meer, 2012], [Yamagishi, Matsumura, 2016], [C., Gergelits, Yamazaki, 2021], [Clark, 2019], [Anzt et al., 2019], [Clark et al., 2010], [Gratton et al., 2020], [Arioli, Duff, 2009], [Hogg, Scott, 2010]

For survey and references, see [Abdelfattah et al., IJHPC, 2021], [Higham and Mary, 2022]

When Can I Use Low Precision?

1. When low accuracy is needed

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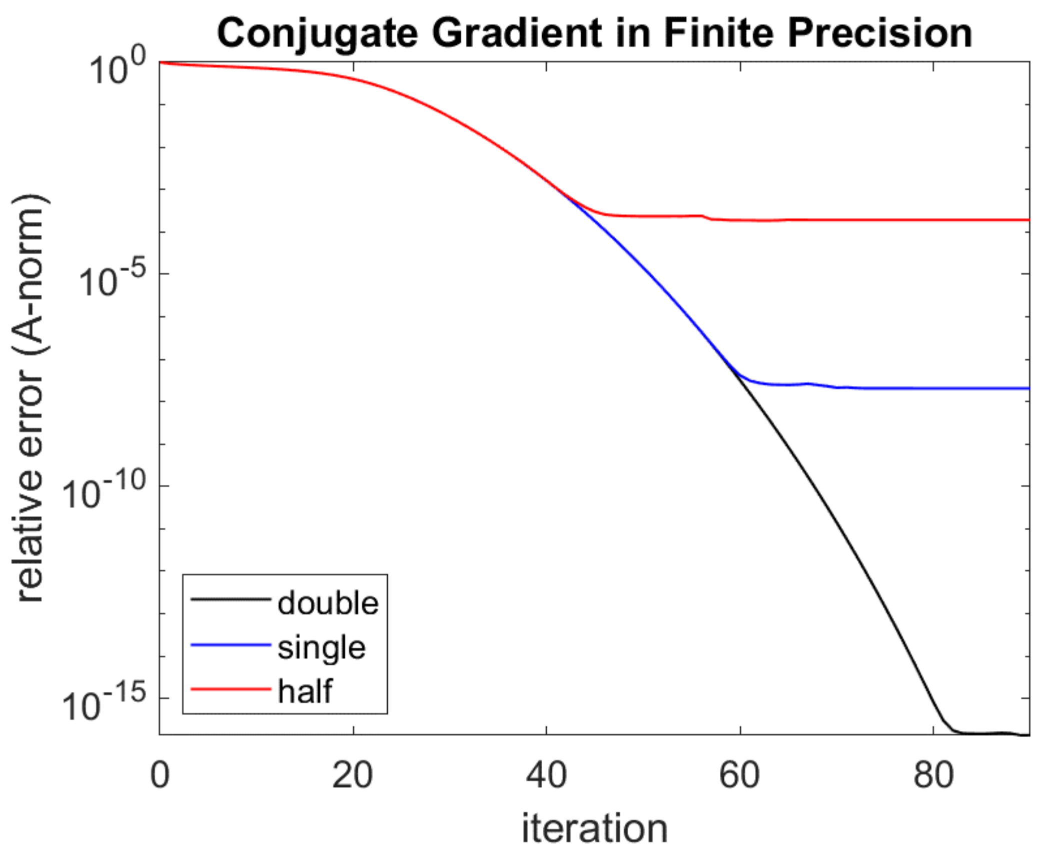
```
A = diag(linspace(.001,1,100));
b = ones(n,1);
```



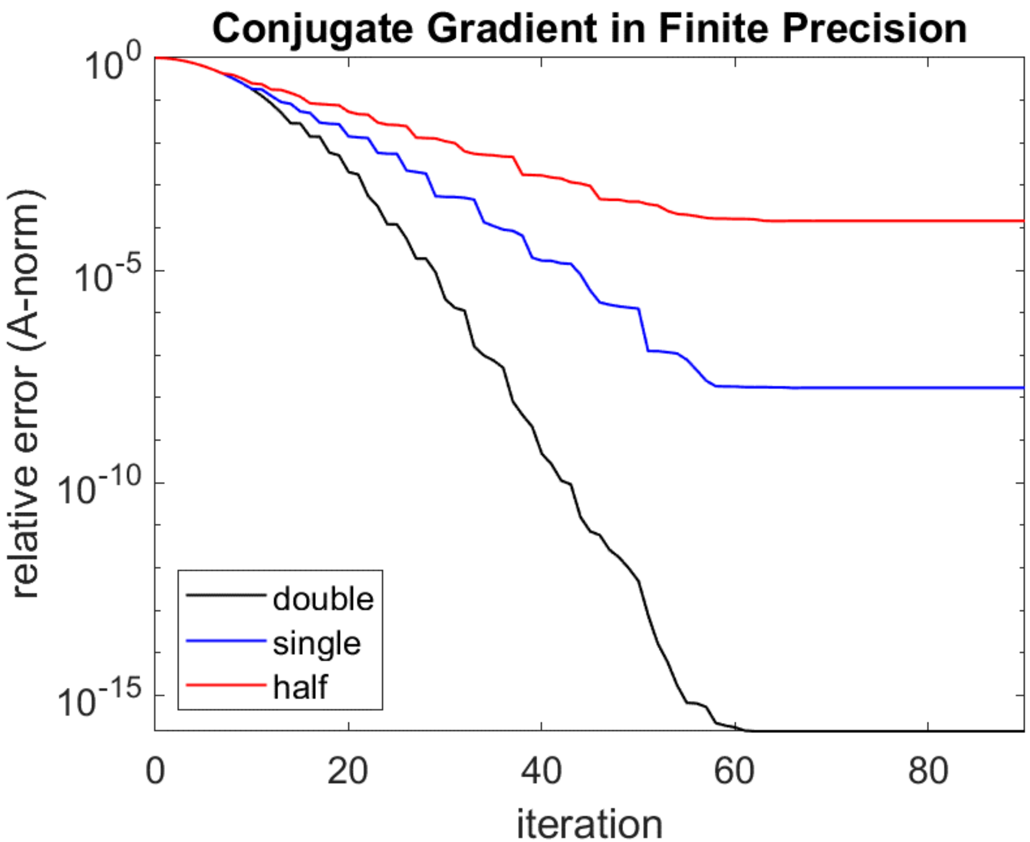
When Can I Use Low Precision?

1. When low accuracy is needed

$n = 100, \lambda_1 = 10^{-3}, \lambda_n = 1$

$\lambda_i = \lambda_1 + \left(\frac{i-1}{n-1}\right)(\lambda_n - \lambda_1)(0.65)^{n-i}, \quad i = 2, \dots, n-1$

```
b = ones(n,1);
```



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2. When a self-correction mechanism is available

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Example: Iterative refinement

Solve $Ax_0 = b$ by LU factorization (in precision $\boldsymbol{u_f}$)

for $i = 0\colon \mathrm{maxit}$

 $r_i = b - Ax_i$ (in precision $\boldsymbol{u_r}$)

 Solve $Ad_i = r_i$ (in precision $\boldsymbol{u_s}$)

 $x_{i+1} = x_i + d_i$ (in precision $\boldsymbol{u}$)

e.g., [Langou et al., 2006], [Arioli and Duff, 2009], [Hogg and Scott, 2010], [Abdelfattah et al., 2016], [C. and Higham, 2018], [Amestoy et al., 2021]

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3. When there are other significant sources of inexactness

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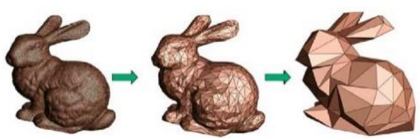
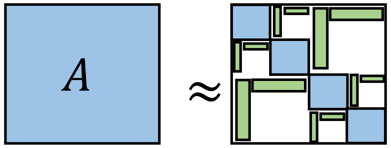
- E.g., reduced models, sparsification, low-rank approximations, randomization

Model Reduction

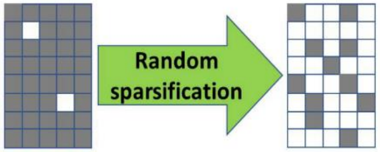


[Schilders, van der Vorst, Rommes, 2008]

Low-rank approximation



Sparsification, randomization



[Sinha, 2018]

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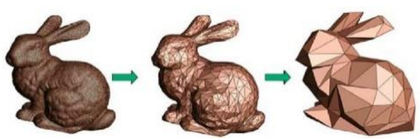

[Schilders, van der Vorst, Rommes, 2008]

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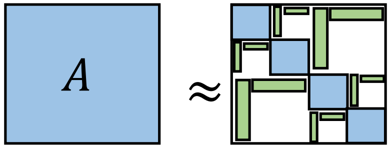

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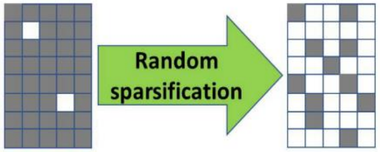

[Sinha, 2018]

Mixed Precision Sparse Approximate Inverse Preconditioners

SPAI Preconditioners

Goal: Construct sparse matrix $M \approx A^{-1}$ (for survey see [Benzi, 2002])

Approach of [Grote, Huckle, 1997]: Construct columns m_k of M dynamically

Given matrix A, initial sparsity structure J, and tolerance $\boldsymbol{\varepsilon}$

For each column k:

- Compute QR factorization of submatrix of A defined by J
- Use QR factorization to solve $\min_{m_k} \|e_k - Am_k\|_2$
- If $\|r_k\|_2 = \|e_k - Am_k\|_2 \leq \boldsymbol{\varepsilon}$
  - break;
- Else
  - add select nonzeros to J, repeat.

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  Else
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Benefits: Highly parallelizable

But construction can still be costly, esp. for large-scale problems

[Gao, Chen, He, 2021], [Chao, 2001], [Benzi, Tůma, 1999], [He, Yin, Gao, 2020]

SPAI Preconditioners in Low Precision

What is the effect of using low precision in SPAI construction?

Notes and assumptions:

- We will assume that the SPAI construction is performed in some precision $\boldsymbol{u_f}$
- We will denote quantities computed in finite precision with hats
- In our application, we want a left preconditioner, so we will run the algorithm on A^T and get M^T.
- We will assume that the QR factorization of the submatrix of A^T is computed fully using HouseholderQR/TSQR

SPAI Preconditioners in Low Precision

Two interesting questions:

1. Assuming we impose no maximum sparsity pattern on $\widehat{M}$, under what constraint on $\boldsymbol{u_f}$ can we guarantee that $\|\hat{r}_k\|_2 \leq \boldsymbol{\varepsilon}$, with $\hat{r}_k = fl_{\boldsymbol{u_f}}(e_k - A^T \widehat{m}_k^T)$ for the computed $\widehat{m}_k^T$?

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2. Assume that when M is computed in exact arithmetic, we quit as soon as $\|r_k\| \leq \boldsymbol{\varepsilon}$. For $\widehat{M}$ computed in precision $\boldsymbol{u_f}$ with the same sparsity pattern as M, what is $\|e_k - A^T \widehat{m}_k^T\|_2$?

SPAI Preconditioning in Low Precision

Using standard rounding error analysis and perturbation results for LS problems, we have

$$\|\hat{r}_k\|_2 \leq n^3 \boldsymbol{u_f} \left\| |e_k| + |A^T| \left|\widehat{m}_k^T\right| \right\|_2.$$

So in order to guarantee we eventually reach a solution with $\|\hat{r}_k\|_2 \leq \boldsymbol{\varepsilon}$, we need

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→ problem must not be so ill-conditioned WRT $\boldsymbol{u_f}$ that we incur an error greater than $\boldsymbol{\varepsilon}$ just computing the residual

SPAI Preconditioning in Low Precision

Can turn this into the looser but more descriptive a priori bound:

$$\mathrm{cond}_2(A^T) \lesssim \boldsymbol{\varepsilon} \boldsymbol{u_f}^{-1},$$

where $\mathrm{cond}_2(A^T) = \| |A^{-T}| |A^T| \|_2$.

SPAI Preconditioning in Low Precision

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Another view: with a given matrix A and a given precision $\boldsymbol{u_f}$, one must set $\boldsymbol{\varepsilon}$ such that

$$\boldsymbol{\varepsilon} \geq \boldsymbol{u_f} \mathrm{cond}_2(A^T).$$

Confirms intuition: **The more approximate the inverse, the lower the precision we can use without noticing it**.

SPAI Preconditioning in Low Precision

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Confirms intuition: **The more approximate the inverse, the lower the precision we can use without noticing it**.

Resulting bounds for $\widehat{M}$:

$$\left\| I - A^T \widehat{M}^T \right\|_F \leq 2\sqrt{n}\boldsymbol{\varepsilon}, \qquad \left\| I - \widehat{M} A \right\|_\infty \leq 2n\boldsymbol{\varepsilon}$$

Second Question

Assume that when M *is computed in exact arithmetic, we quit as soon as* $\|r_k\| \leq \boldsymbol{\varepsilon}$*. For* $\widehat{M}$ *computed in precision* $\boldsymbol{u_f}$ *with the same sparsity pattern as* M*, what is* $\left\|e_k - A^T \widehat{m}_k^T\right\|_2$*?*

Second Question

Assume that when M is computed in exact arithmetic, we quit as soon as $\|r_k\| \leq \boldsymbol{\varepsilon}$. For $\widehat{M}$ computed in precision $\boldsymbol{u_f}$ with the same sparsity pattern as M, what is $\left\|e_k - A^T \widehat{m}_k^T\right\|_2$?

In this case, we obtain the bound

$$\left\|I - \widehat{M}A\right\|_\infty \leq n\left(\boldsymbol{\varepsilon} + n^{7/2}\boldsymbol{u_f}\kappa_\infty(A)\right).$$

→ If $\kappa_\infty(A) \gg \boldsymbol{\varepsilon}\boldsymbol{u_f}^{-1}$, then computed $\widehat{M}$ with same sparsity structure as M can be of much lower quality.

Krylov-Based Iterative Refinement

Solve $Ax_0 = b$ by LU factorization (in precision $\boldsymbol{u_f}$)

for $i = 0$: maxit

$r_i = b - Ax_i$ (in precision $\boldsymbol{u_r}$)

Solve $Ad_i = r_i$ (in precision $\boldsymbol{u_s}$)

$x_{i+1} = x_i + d_i$ (in precision $\boldsymbol{u}$)

Krylov-Based Iterative Refinement

GMRES-IR [C. and Higham, SISC 39(6), 2017]

To compute the updates d_i, apply GMRES to $\overbrace{\hat{U}^{-1}\hat{L}^{-1}A}^{\tilde{A}}d_i = \overbrace{\hat{U}^{-1}\hat{L}^{-1}r_i}^{\tilde{r}_i}$

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For related work, see references in [Higham, Mary, 2022], [Vieuble, 2022]

GMRES-IR with Inexact Preconditioners

- Most existing analyses of GMRES-IR assume we use full LU factors
- In practice, often want to use approximate preconditioners (ILU, SPAI, etc.)

- [Amestoy et al., 2022]
  - Analysis of **block low-rank (BLR) LU** within GMRES-IR
  - Analysis of use of **static pivoting** in LU within GMRES-IR

- [C., Khan, 2023]
  - Analysis of **sparse approximate inverse (SPAI) preconditioners** within GMRES-IR

SPAI-GMRES-IR

SPAI-GMRES-IR [C. and Khan, SISC 45(3), 2023]

To compute the updates d_i, apply GMRES to $\overbrace{\widehat{M}A}^{\tilde{A}} d_i = \overbrace{\widehat{M} r_i}^{\tilde{r}_i}$

Compute SPAI $\widehat{M}$; solve $\widehat{M}Ax_0 = \widehat{M}b$ (in precision $\boldsymbol{u_f}$)

for $i = 0\!:\!\text{maxit}$

$r_i = b - Ax_i$ (in precision $\boldsymbol{u_r}$)

Solve $Ad_i = r_i$ via GMRES on $\widehat{M}Ad_i = \widehat{M}r_i$ (in precision $\boldsymbol{u_s}$)

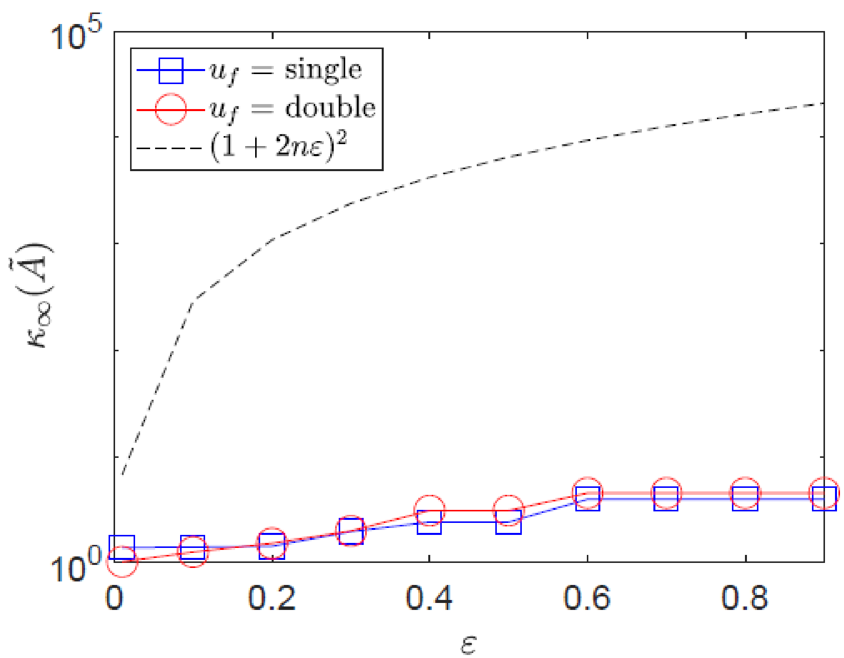
$x_{i+1} = x_i + d_i$ (in precision $\boldsymbol{u}$)

Low Precision SPAI within GMRES-IR

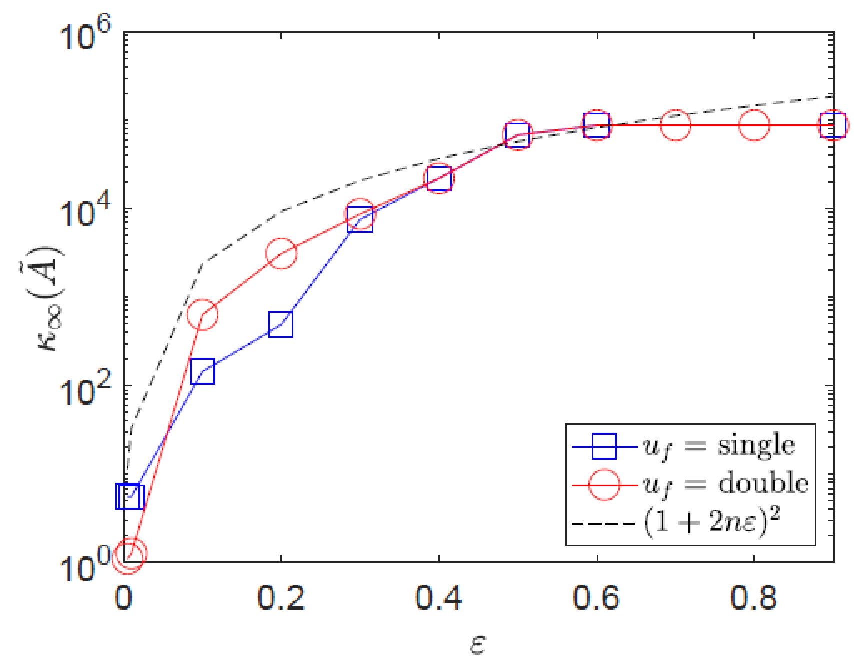
Using $\widehat{M}$ computed in precision $\boldsymbol{u_f}$, for the preconditioned system $\tilde{A} = \widehat{M}A$,

$$\kappa_\infty(\tilde{A}) \lesssim (1 + 2n\boldsymbol{\varepsilon})^2.$$

steam3



saylr1



Low Precision SPAI within GMRES-IR

To guarantee that both SPAI construction will complete and the GMRES-based iterative refinement scheme will converge, we must have roughly

$$n\boldsymbol{u_f}\,\mathrm{cond}_2(A^T) \lesssim n\boldsymbol{\varepsilon} \lesssim \boldsymbol{u}^{-1/2}.$$

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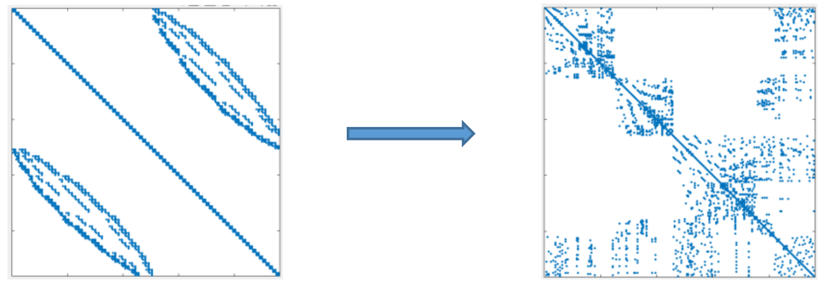
If $\boldsymbol{\varepsilon}$ satisfies these constraints, then the constraints on condition number for forward and backward errors to converge are the same as for GMRES-IR with full LU factorization.

Compared to GMRES-IR with full LU factorization, in general expect **slower convergence, but much sparser preconditioner**.

SPAI-GMRES-IR Example

Matrix: steam1, $n = 240$, nnz $= 2{,}248$, $\kappa_\infty(A) = 3 \cdot 10^7$, cond$(A^T) = 3 \cdot 10^3$



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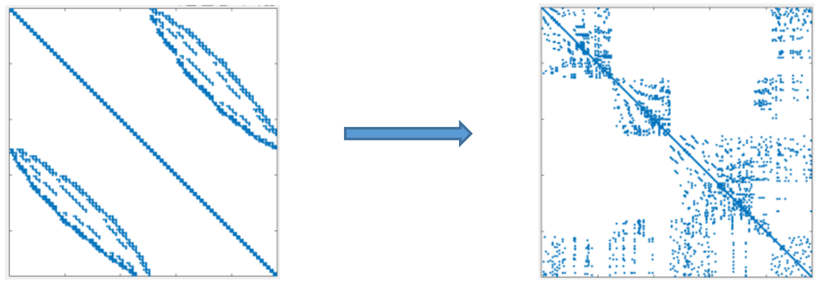
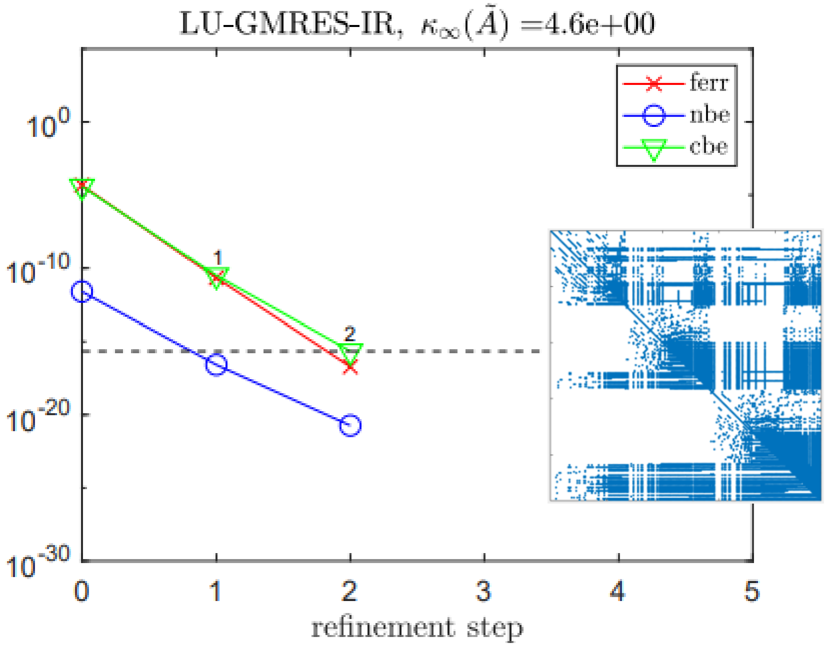


$(\boldsymbol{u_f}, \boldsymbol{u}, \boldsymbol{u_r})$ = (single, double, quad)



nnz$(L + U)$ = 13,765

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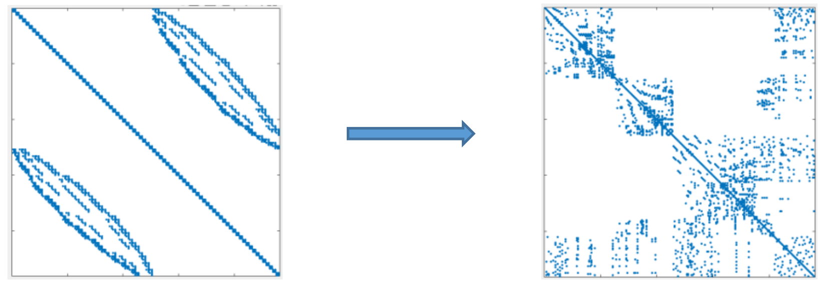

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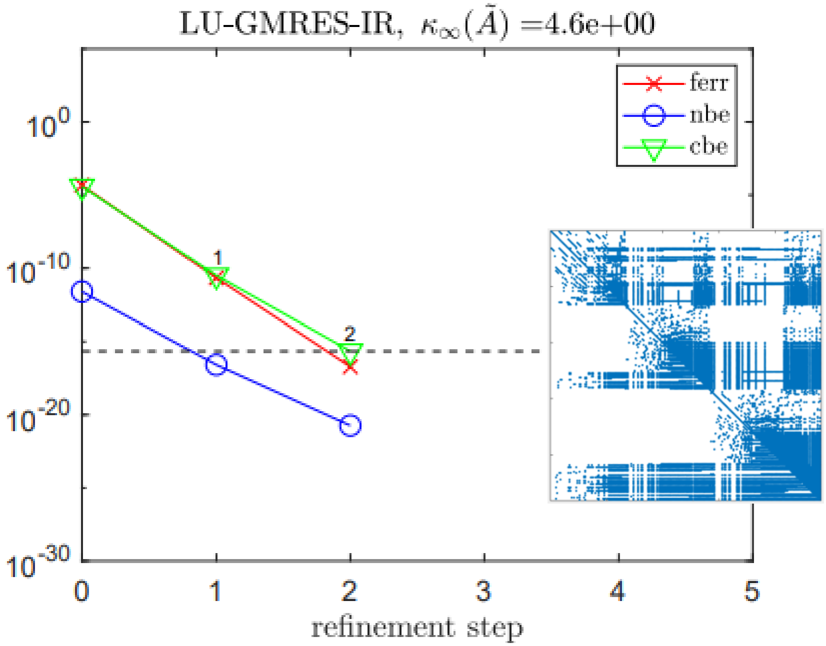

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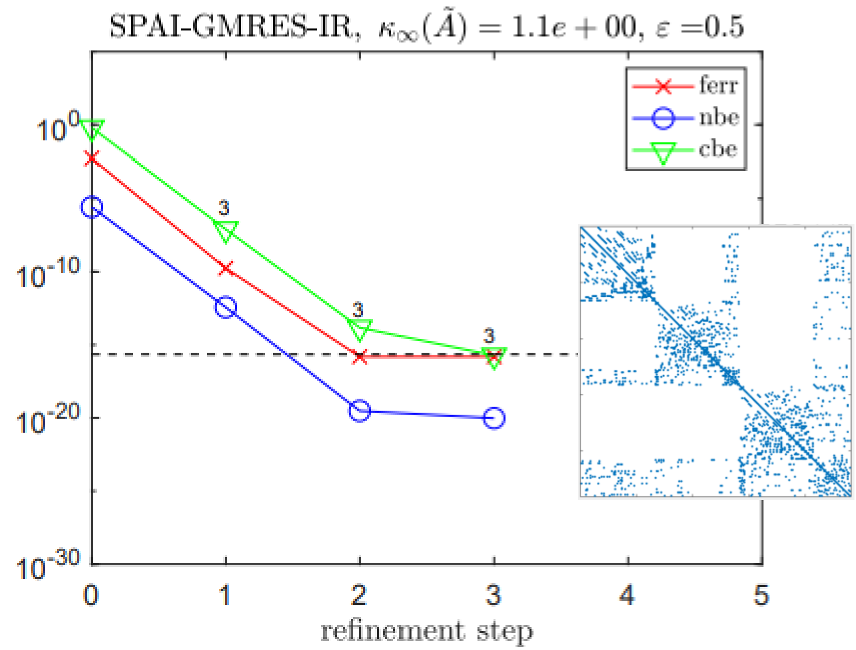

nnz$(M) = 2{,}248$

Ongoing and Future Work

- Incorporate mixed-precision storage of $\widehat{M}$ and **adaptive-precision SpMV** to apply $\widehat{M}$ using the work of [Graillat et al., 2022]

- Theoretical analysis of **incomplete factorization preconditioners** in mixed precision (with J. Scott and M. Tůma)
    - Experimental work shows that half precision works well in practice [Scott, Tůma, 2023]

Randomized Preconditioners for GMRES-Based Least Squares Iterative Refinement

Least Squares Problems

- Want to solve

$$\min_x \|b - Ax\|_2$$

where $A \in \mathbb{R}^{m \times n}$ $(m > n)$ has rank n

- Commonly solved using QR factorization:

$$A = QR = [Q_1, Q_2] \begin{bmatrix} U \\ 0 \end{bmatrix}$$

where Q is an $m \times m$ orthogonal matrix and U is upper triangular.

$$x = U^{-1} Q_1^T b, \qquad \|b - Ax\|_2 = \left\|Q_2^T b\right\|_2$$

- As in linear system case, for ill-conditioned problems, iterative refinement often needed to improve accuracy and stability

Least Squares Iterative Refinement

- For inconsistent systems, must simultaneously refine both solution and residual
- (Björck, 1967): Least squares problem can be written as a linear system with square matrix of size $(m+n)$:

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

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- Refinement proceeds as follows:

1. Compute "residuals"

$$\begin{bmatrix} f_i \\ g_i \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} - \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r_i \\ x_i \end{bmatrix} = \begin{bmatrix} b - r_i - Ax_i \\ -A^T r_i \end{bmatrix}$$

2. Solve for corrections

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \Delta r_i \\ \Delta x_i \end{bmatrix} = \begin{bmatrix} f_i \\ g_i \end{bmatrix}$$

3. Update "solution":

$$\begin{bmatrix} r_{i+1} \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} r_i \\ x_i \end{bmatrix} + \begin{bmatrix} \Delta r_i \\ \Delta x_i \end{bmatrix}$$

GMRES-LSIR

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[C., Higham, Pranesh, 2020]: Compute QR factorization in $\boldsymbol{u_f}$, use as preconditioner for GMRES

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(in precision $\boldsymbol{u_r}$)

2. Solve for corrections

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \Delta r_i \\ \Delta x_i \end{bmatrix} = \begin{bmatrix} f_i \\ g_i \end{bmatrix}$$

via preconditioned GMRES (in precision $\boldsymbol{u_s}$)

3. Update "solution":

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(in precision $\boldsymbol{u}$)

GMRES-LSIR Analysis

- Using the preconditioner

$$M = \begin{bmatrix} \alpha I & \hat{Q}_1 \hat{R} \\ \hat{R}^T \hat{Q}_1^T & 0 \end{bmatrix}$$

we can prove that for the left-preconditioned system,

$$\kappa\left(M^{-1}\tilde{A}\right) \le \left(1 + \boldsymbol{u_f} c\, \kappa(A)\right)^2$$

where $c = O(m^2)$.

[C., Higham, Pranesh, SISC 2020]

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- So for GMRES-based LSIR, expect convergence of forward error when $\kappa_\infty(A) < \boldsymbol{u}^{-1/2}\boldsymbol{u_f}^{-1}$.

[C., Higham, Pranesh, SISC 2020]

- Using the preconditioner

$$M = \begin{bmatrix} \alpha I & \hat{Q}_1 \hat{R} \\ \hat{R}^T \hat{Q}_1^T & 0 \end{bmatrix}$$

Can we use other preconditioners?

we can prove that for the left-preconditioned system,

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where $c = O(m^2)$.

- So for GMRES-based LSIR, expect convergence of forward error when $\kappa_\infty(A) < \boldsymbol{u}^{-1/2}\boldsymbol{u_f}^{-1}$.

[C., Higham, Pranesh, SISC 2020]

Randomized Preconditioning for LS

"Sketch-and-precondition" [Rokhlin, Tygert, 2008]:

1. Randomly sketch A

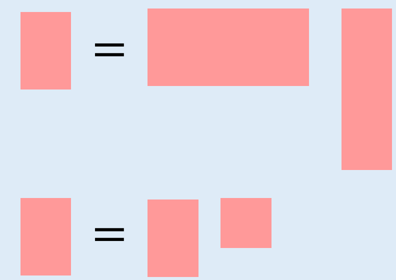
$$S = \Omega A, \text{ where } S \in \mathbb{R}^{s\times m},\ s \geq n$$

2. Compute economic QR

$$S = QR$$

3. Solve via LSQR preconditioned with R

$$\min_y \|b - AR^{-1}y\|_2, \quad \text{where } y = Rx$$



[Avron, Maymounkov, Toledo, 2010]: Efficient implementation (Blendenpik) in one precision

Randomized Preconditioning for LS

$\boldsymbol{u} = \boldsymbol{u_{QR}} \leq \boldsymbol{u_s}$

"Sketch-and-precondition" [Rokhlin, Tygert, 2008]:

1. Randomly sketch A

   $S = \Omega A$, where $S \in \mathbb{R}^{s \times m}$, $s \geq n$ (in precision $\boldsymbol{u_s}$)

2. Compute economic QR

   $S = QR$ (in precision $\boldsymbol{u_{QR}}$)

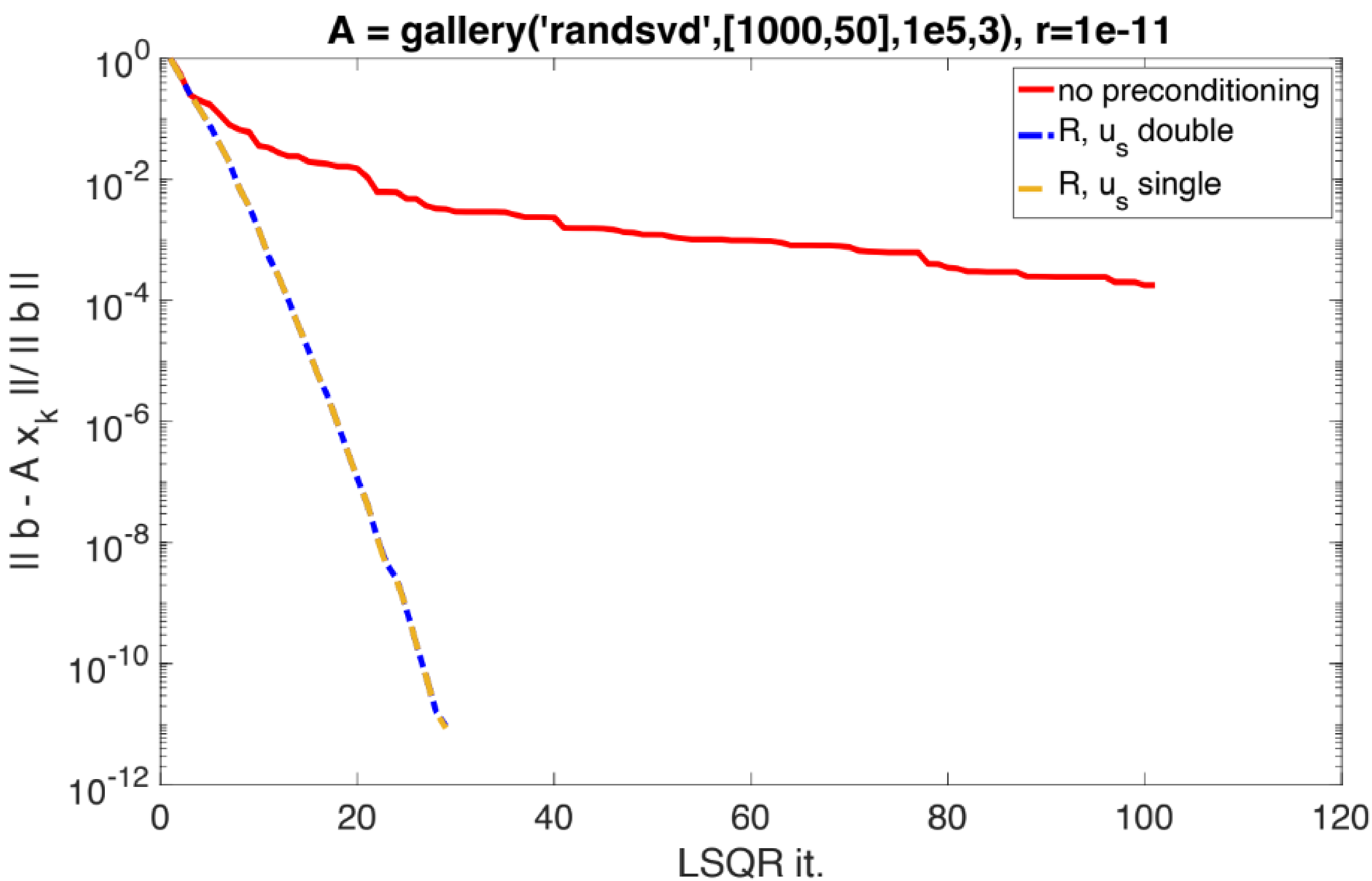
3. Solve via LSQR preconditioned with R

   $\min_y \|b - AR^{-1}y\|_2$, where $y = Rx$ (in precision $\boldsymbol{u}$)

[Avron, Maymounkov, Toledo, 2010]: Efficient implementation (Blendenpik) in one precision

[Georgiou, Boutsikas, Drineas, Anzt, 2023]: Experimental results that show R can be computed in mixed precision

$u = u_{QR} =$ double



A = gallery('randsvd',[1000,50],1e5,3), r=1e-11
no preconditioning
R, u_s double
R, u_s single
|| b - A x_k ||/ || b ||
LSQR it.
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
0
20
40
60
80
100
120

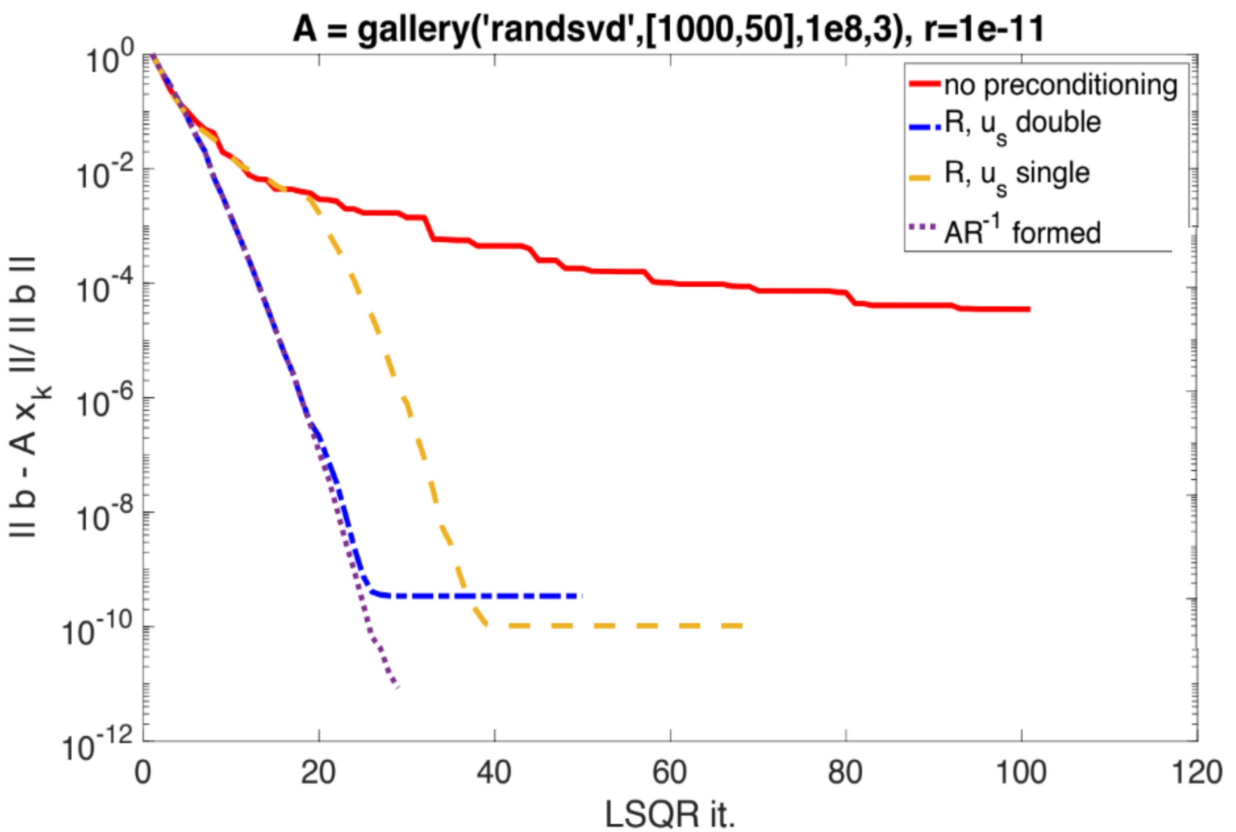
Randomized Preconditioning

"Sketch-and-apply" [Meier, Nakatsukasa, Townsend, Webb, 2023]

1. Compute R as in [Rokhlin, Tygert, 2008]
2. Explicitly form preconditioned matrix
$$Y = AR^{-1}$$
3. Solve via (unpreconditioned) LSQR
$$\min_z \|b - Yz\|_2$$
4. Recover x
$$Rx = z$$

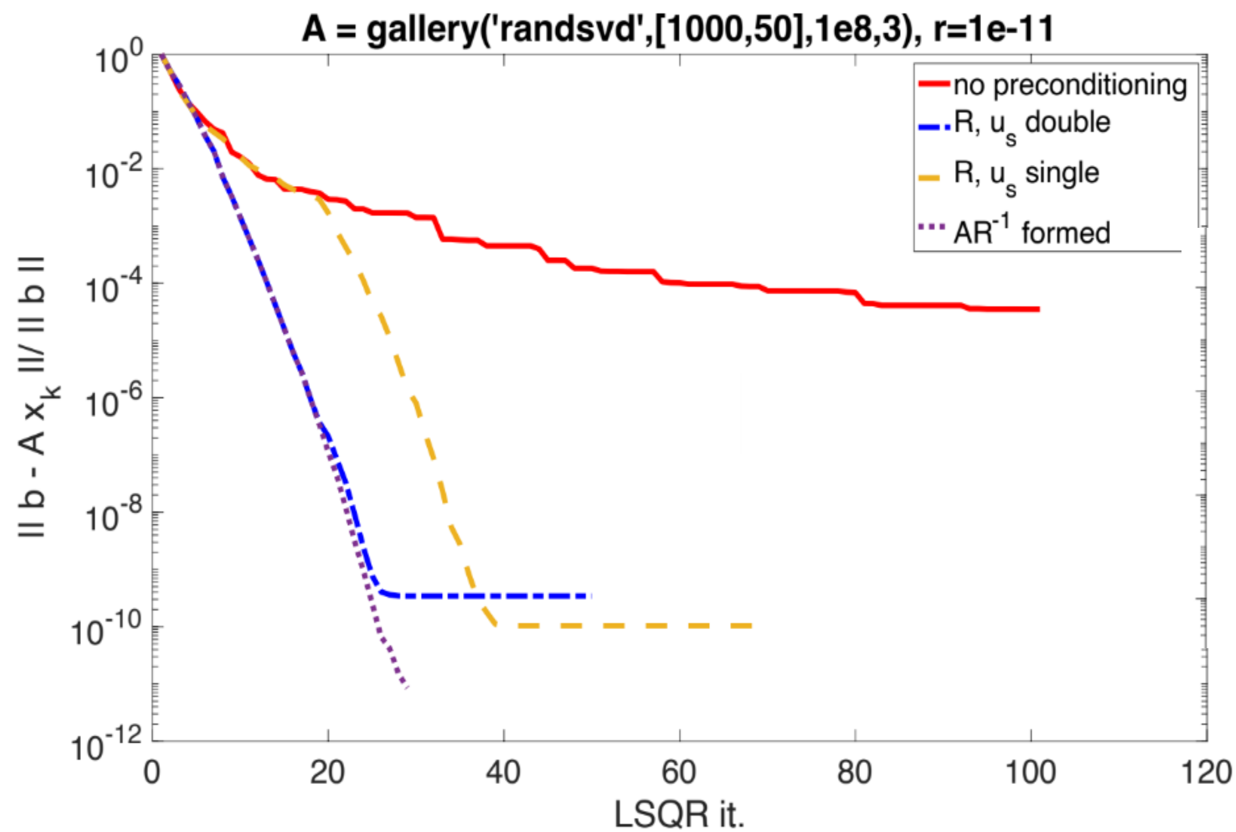
Example

$\boldsymbol{u} = \boldsymbol{u_{QR}} =$ double



Example

$\boldsymbol{u} = \boldsymbol{u_{QR}}$ = double



Relative forward error:

R, u_s double: 4×10^{-8}
R, u_s double: 3×10^{-8}
Formed AR^{-1}: 2×10^{-8}

“Sketch-and-Precondition” GMRES-LSIR

Compute $\hat{R}$ factor of QR decomposition of randomly sketched A using precision $\boldsymbol{u_s}$ (sketching step) and $\boldsymbol{u_{QR}}$ (QR step).

"Sketch-and-Precondition" GMRES-LSIR

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Solve $\min_x \|b - Ax\|_2$ via LSQR preconditioned with $\hat{R}$ in precision $\boldsymbol{u}$ to get initial solution x_0 and residual r_0.

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for $i = 0, \dots,$ until convergence

Compute residual $\begin{bmatrix} f_i \\ g_i \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} - \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r_i \\ x_i \end{bmatrix}$ and $h_i = \hat{R}^{-T} g_i$ in precision $\boldsymbol{u_r}$.

Solve via FGMRES in (effective) precision $\boldsymbol{u_s}$:

$$\begin{bmatrix} I & 0 \\ 0 & \hat{R}^{-T} \end{bmatrix} \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \hat{R}^{-1} \end{bmatrix} \begin{bmatrix} \delta r_i \\ \delta z_i \end{bmatrix} = \begin{bmatrix} f_i \\ h_i \end{bmatrix},$$

where $\hat{R} \delta x_i = \delta z_i$.

Update in precision $\boldsymbol{u}$:

$$\begin{bmatrix} r_{i+1} \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} r_i \\ x_i \end{bmatrix} + \begin{bmatrix} \delta r_i \\ \delta x_i \end{bmatrix}$$

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[C., Daužikaitė, 2023]: Analysis of four-precision split-preconditioned FGMRES

"Sketch-and-Precondition" GMRES-LSIR

Theoretical analysis suggests how to choose precisions:

- For generating preconditioner, $\boldsymbol{u_s} \approx \boldsymbol{u_{QR}}$ (although $\boldsymbol{u_{QR}} < \boldsymbol{u_s}$ is inexpensive and may help avoid overflow)

- For FGMRES, apply left preconditioner and matrix to a vector in precision $\leq \boldsymbol{u}$ (can be less careful with right preconditioner)

“Sketch-and-Apply” GMRES-LSIR

Compute $\hat{R}$ factor of QR decomposition of randomly sketched A using precision $\boldsymbol{u_s}$ (sketching step) and $\boldsymbol{u_{QR}}$ (QR step).

“Sketch-and-Apply” GMRES-LSIR

Compute $\hat{R}$ factor of QR decomposition of randomly sketched A using precision $\boldsymbol{u_s}$ (sketching step) and $\boldsymbol{u_{QR}}$ (QR step).

Form $Y = A\hat{R}^{-1}$ in precision $\boldsymbol{u_Y}$.

"Sketch-and-Apply" GMRES-LSIR

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Solve $\min_z \|b - Yz\|_2$ via LSQR in precision $\boldsymbol{u}$ and solve $Rx = z$ in precision $\boldsymbol{u_x}$ to get initial solution x_0 and residual r_0.

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Solve via unpreconditioned GMRES in precision $\boldsymbol{u}$:

$$\begin{bmatrix} I & Y \\ Y^T & 0 \end{bmatrix} \begin{bmatrix} \delta r_i \\ \delta z_i \end{bmatrix} = \begin{bmatrix} f_i \\ h_i \end{bmatrix}$$

Solve $\hat{R}\delta x_i = \delta z_i$ in precision $\boldsymbol{u_x}$.

Update in precision $\boldsymbol{u}$:

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"Sketch-and-Apply" GMRES-LSIR

Theoretical analysis suggests how to choose precisions:

- For generating preconditioner, $\boldsymbol{u_s} \approx \boldsymbol{u_{QR}}$ (although $\boldsymbol{u_{QR}} < \boldsymbol{u_s}$ is inexpensive and may help avoid overflow)

- Triangular solves: Want $\boldsymbol{u_x}\kappa(A) < 1$
- GMRES: Want $\boldsymbol{u}\kappa(A)\kappa(Y) < 1$
- Forming Y: Want $\boldsymbol{u_Y}\kappa(A)^2\kappa(Y) < 1$

Ongoing work: Collaboration on high-performance implementation with V. Georgiou and H. Anzt

Mixed Precision Randomized Nyström Approximation

Randomized Nyström Approximation

Want to compute a rank-k approximation $A \approx U\Theta U^T$ via the randomized Nyström method.

Nyström approximation:

$$A_N = (A\Omega)(\Omega^T A\Omega)^\dagger (A\Omega)^T$$

where Ω is an $n \times k$ sampling matrix

Many applications: approximation of kernel matrices, spectral limited memory preconditioners, etc.

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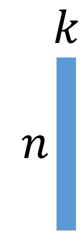
Many applications: approximation of kernel matrices, spectral limited memory preconditioners, etc.

In the case that A is very large, matrix-matrix products with A are the bottleneck.

→ Can use single-pass version of the Nyström method [Tropp et al., 2017].

Single-Pass Nyström Approximation

Given sym. PSD matrix A, target rank k



$G = \text{randn}(n, k)$

$[Q, \sim] = \text{qr}(G, 0)$

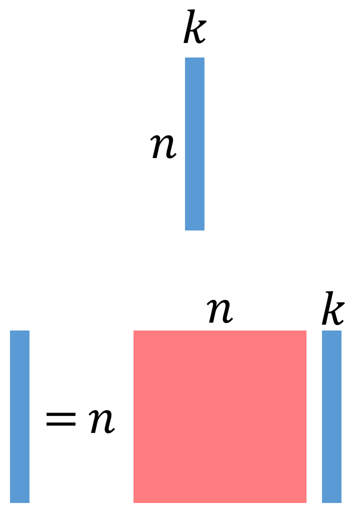
Single-Pass Nyström Approximation

Given sym. PSD matrix A, target rank k

$G = \text{randn}(n, k)$

$[Q, \sim] = \text{qr}(G, 0)$

$\boldsymbol{Y = AQ}$



Single-Pass Nyström Approximation

Given sym. PSD matrix A, target rank k

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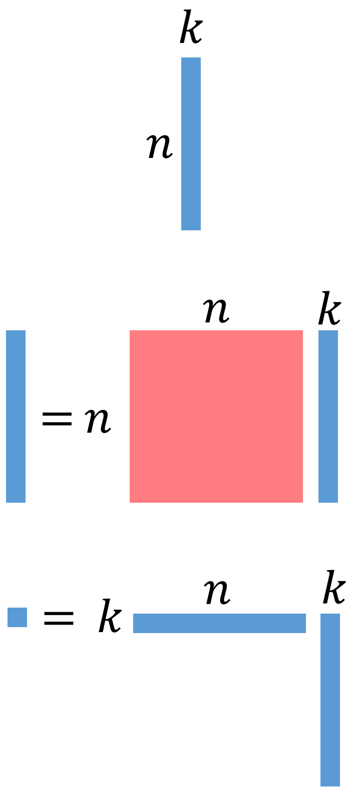
$\boldsymbol{Y = AQ}$

Compute shift ν; $Y_\nu = Y + \nu Q$

$B = Q^T Y_\nu$



Single-Pass Nyström Approximation

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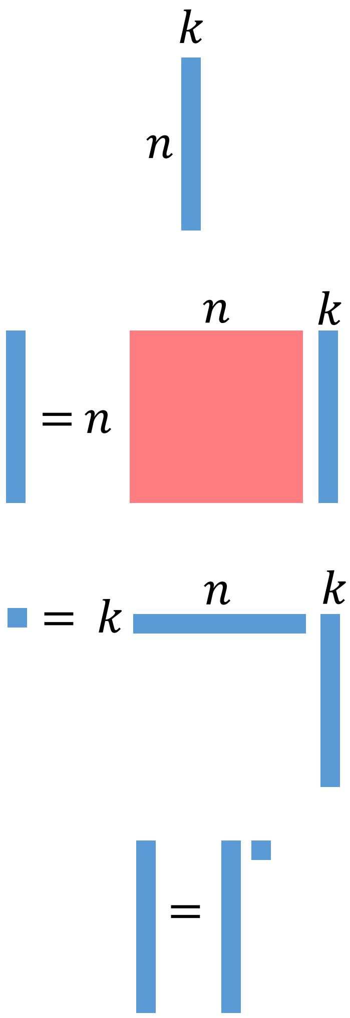
Compute shift ν; $Y_\nu = Y + \nu Q$

$B = Q^T Y_\nu$

$C = \text{chol}((B + B^T)/2)$

Solve $F = Y_\nu / C$



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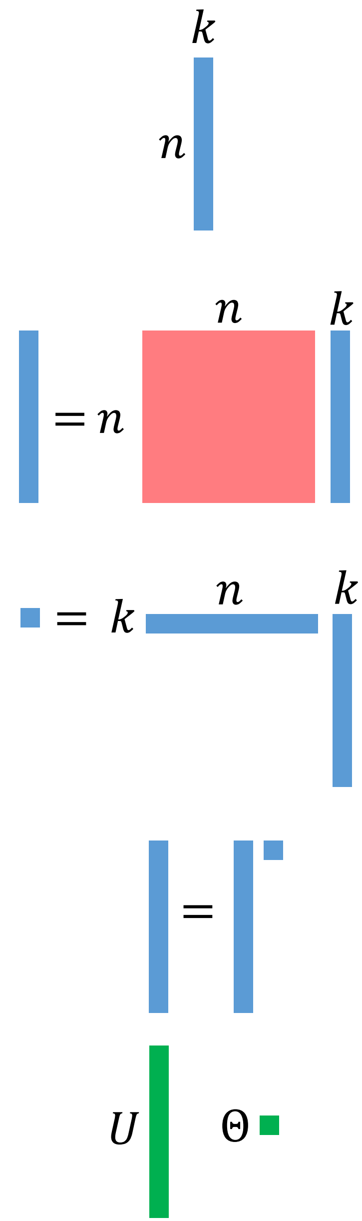
Solve $F = Y_\nu / C$

$[U, \Sigma, \sim] = \mathrm{svd}(F, 0)$

$\Theta = \max(0, \Sigma^2 - \nu I)$



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Solve $F = Y_\nu / C$

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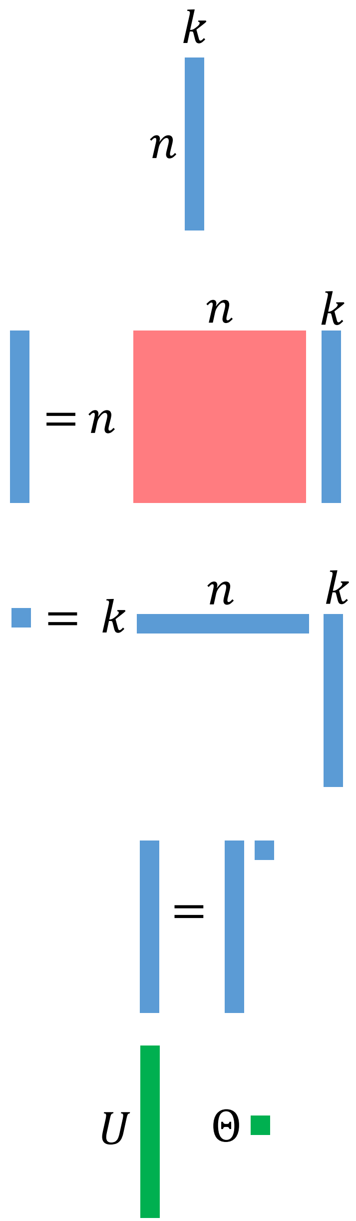
$\Theta = \max(0, \Sigma^2 - \nu I)$

Can we further reduce the cost of the matrix-matrix product with A by using low precision?



Single-Pass Nyström Approximation

Given sym. PSD matrix A, target rank k

$\boldsymbol{u} \ll \boldsymbol{u_p}$

$G = \text{randn}(n, k)$

$[Q, \sim] = \text{qr}(G, 0)$ (precision $\boldsymbol{u}$)

$\boldsymbol{Y = AQ}$ (precision $\boldsymbol{u_p}$)

Compute shift ν; $Y_\nu = Y + \nu Q$ (precision $\boldsymbol{u}$)

$B = Q^T Y_\nu$ (precision $\boldsymbol{u}$)

$C = \text{chol}((B + B^T)/2)$ (precision $\boldsymbol{u}$)

Solve $F = Y_\nu / C$ (precision $\boldsymbol{u}$)

$[U, \Sigma, \sim] = \text{svd}(F, 0)$ (precision $\boldsymbol{u}$)

$\Theta = \max(0, \Sigma^2 - \nu I)$ (precision $\boldsymbol{u}$)

Error Bounds

$$\left\|A - \hat{A}_N\right\|_2 = \left\|A - A_N + A_N - \hat{A}_N\right\|_2 \leq \|A - A_N\|_2 + \left\|A_N - \hat{A}_N\right\|_2$$

exact Nyström approximation

Nyström approximation computed in finite precision

Error Bounds

$$\left\|A - \hat{A}_N\right\|_2 = \left\|A - A_N + A_N - \hat{A}_N\right\|_2 \leq \underbrace{\|A - A_N\|_2}_{\text{exact approximation error}} + \underbrace{\left\|A_N - \hat{A}_N\right\|_2}_{\text{finite precision error}}$$

Error Bounds

$$\left\|A - \hat{A}_N\right\|_2 = \left\|A - A_N + A_N - \hat{A}_N\right\|_2 \leq \underbrace{\|A - A_N\|_2}_{\text{exact approximation error}} + \underbrace{\left\|A_N - \hat{A}_N\right\|_2}_{\text{finite precision error}}$$

Deterministic bound [Gittens, Mahoney, 2016]

Expected value bound [Frangella, Tropp, Udell, 2021]

Error Bounds

$$\left\|A - \hat{A}_N\right\|_2 = \left\|A - A_N + A_N - \hat{A}_N\right\|_2 \leq \underbrace{\|A - A_N\|_2}_{\text{exact approximation error}} + \underbrace{\left\|A_N - \hat{A}_N\right\|_2}_{\text{finite precision error}}$$

[C., Daužickaitė, 2022]: With failure probability at most $e^{-t^2/2} + c_1\alpha$,

$$\left\|A_N - \hat{A}_N\right\|_2 \lesssim \alpha^{-1} n^{1/2} k \left(n^{1/2} + k^{1/2} + t\right)^2 \boldsymbol{u_p} \|A\|_2 \kappa(A_k)$$

where A_k is the best rank-k approximation of A.

Error Bounds

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Interpretation: Likely that $\left\|A_N - \hat{A}_N\right\|_2 \gtrsim \|A - A_N\|_2$ when

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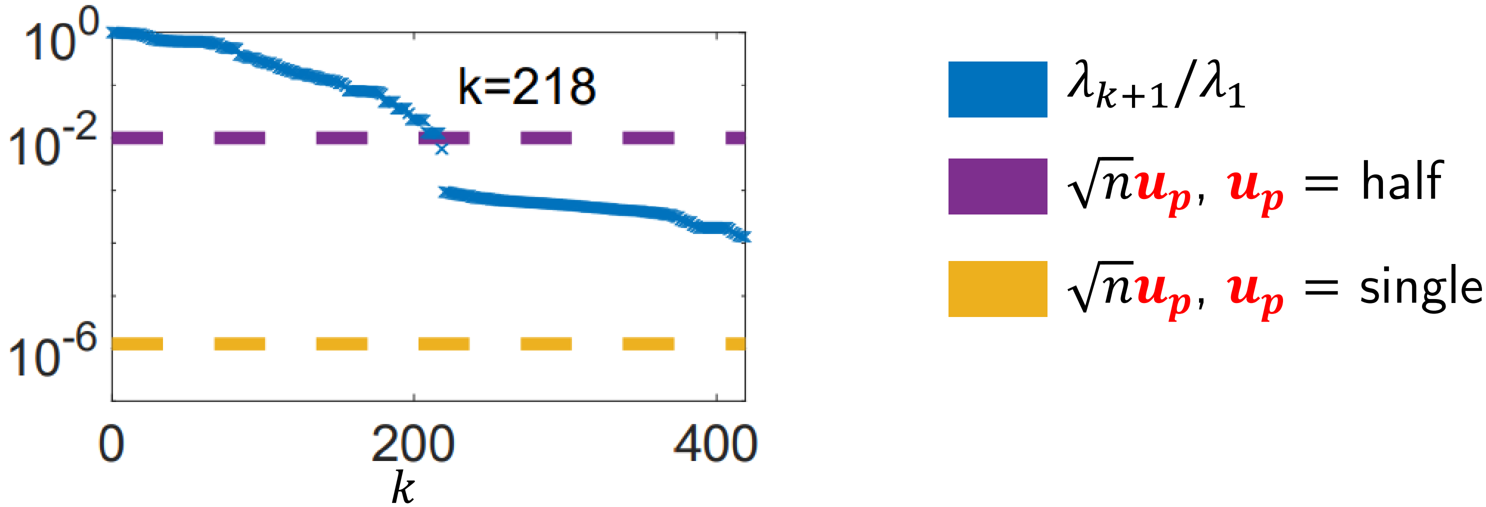
The worse the low-rank representation, the lower the precision we can use!

Numerical Experiment

Matrix: bcsstm07, $n = 420$



https://github.com/dauzickaite/mpNystrom

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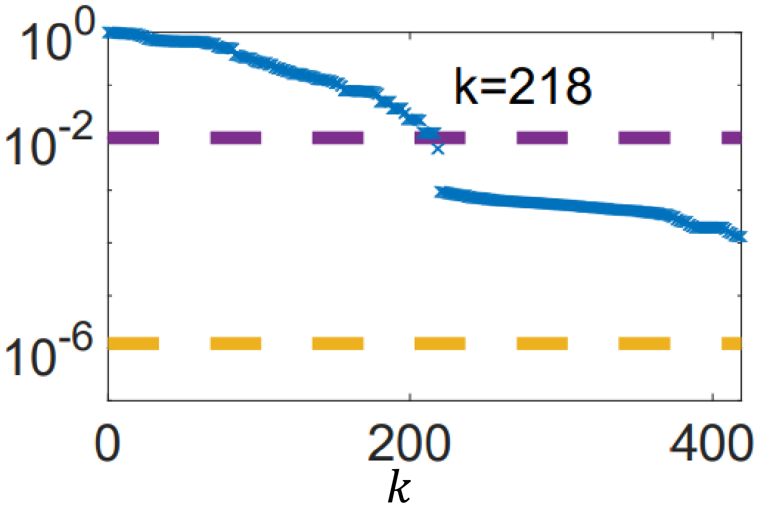
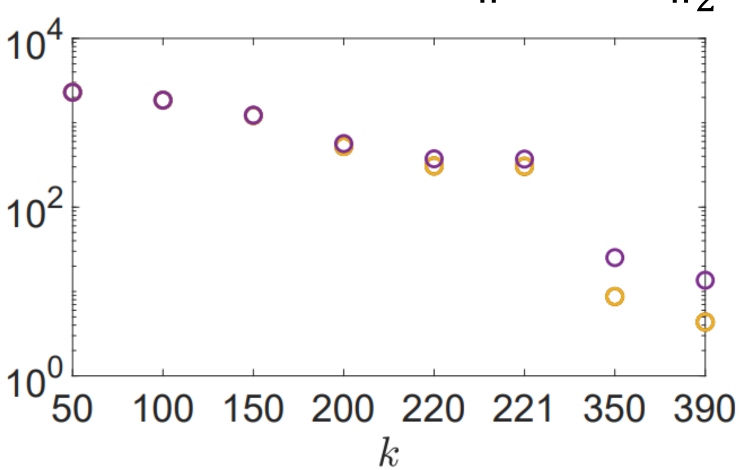


- λ_{k+1}/λ_1
- $\sqrt{n}\boldsymbol{u_p}$, $\boldsymbol{u_p}$ = half
- $\sqrt{n}\boldsymbol{u_p}$, $\boldsymbol{u_p}$ = single

mean total error, $\|A - \hat{A}_N\|_2$



- exact
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- $\boldsymbol{u_p}$ = single, $\boldsymbol{u}$ = double
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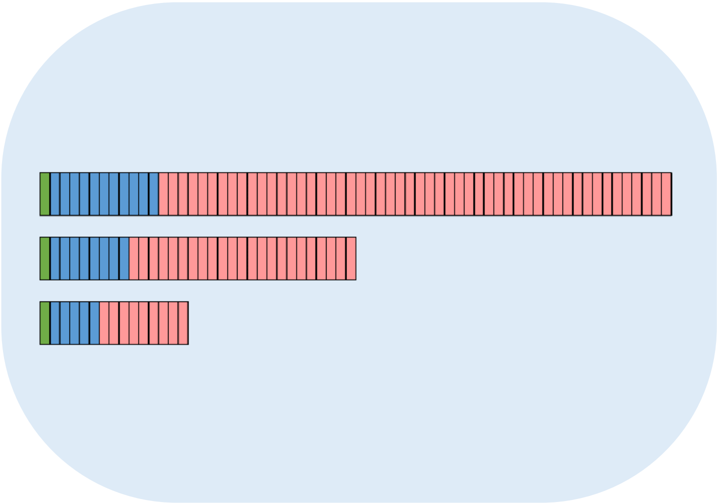
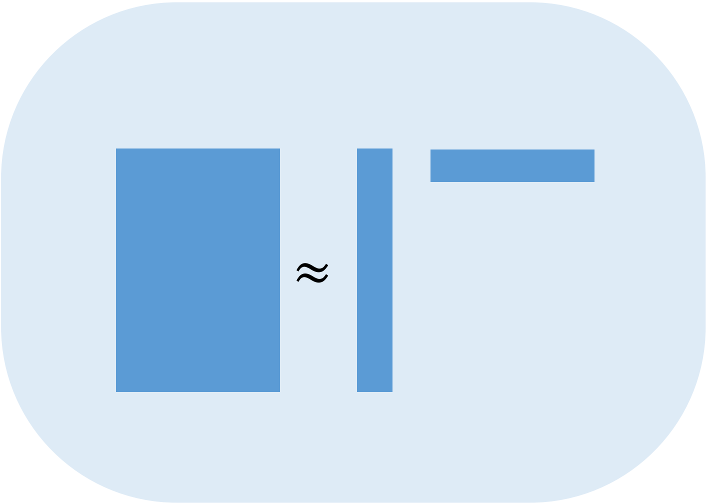
Summary and Takeaway





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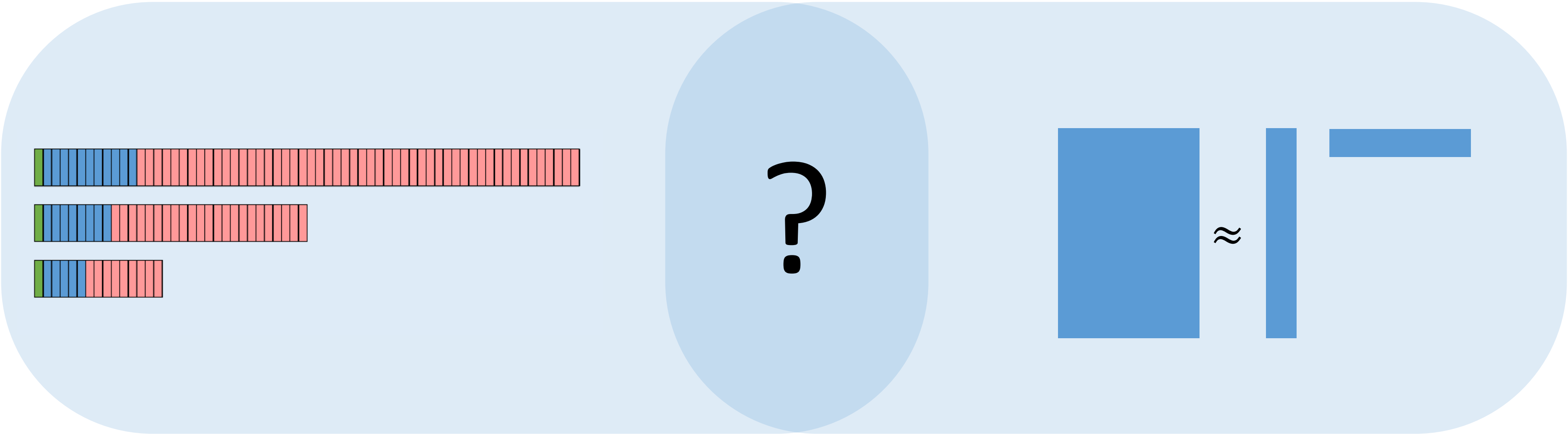

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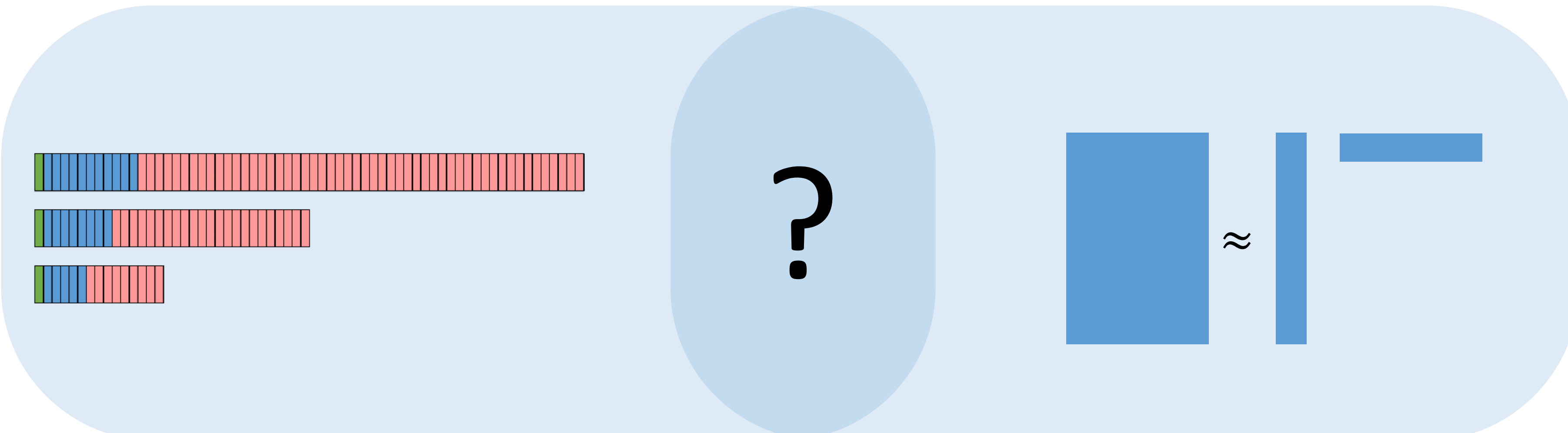


Where can ***you*** use mixed or low precision?

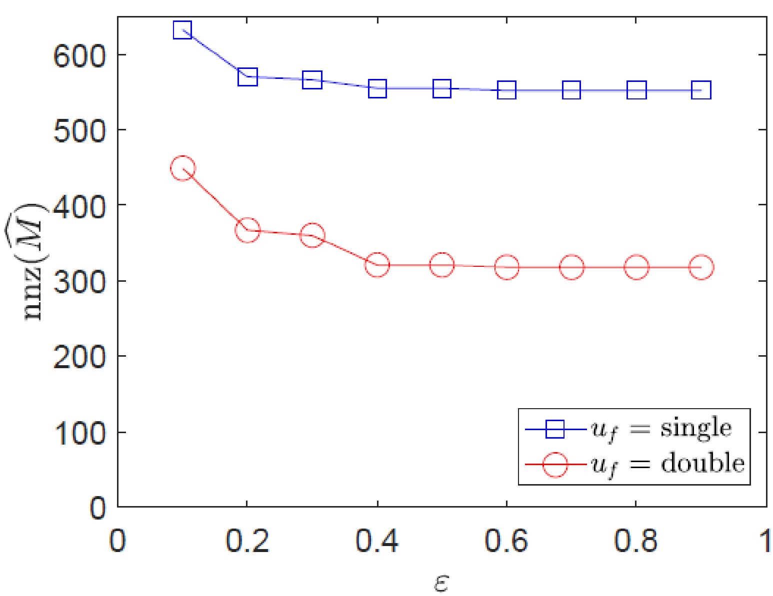
Thank You!

carson@karlin.mff.cuni.cz
www.karlin.mff.cuni.cz/~carson/

Size of SPAI Preconditioner in Low Precision

How does precision used affect the number of nonzeros in $\widehat{M}$?

steam3



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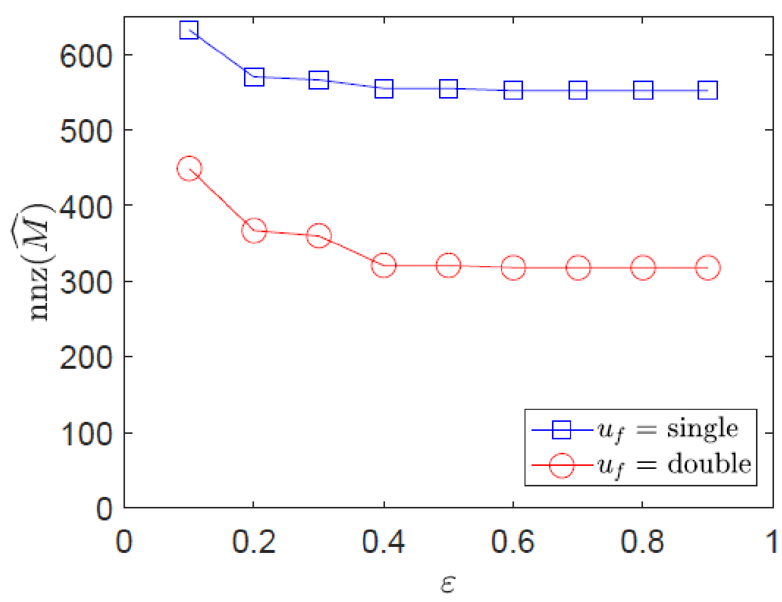
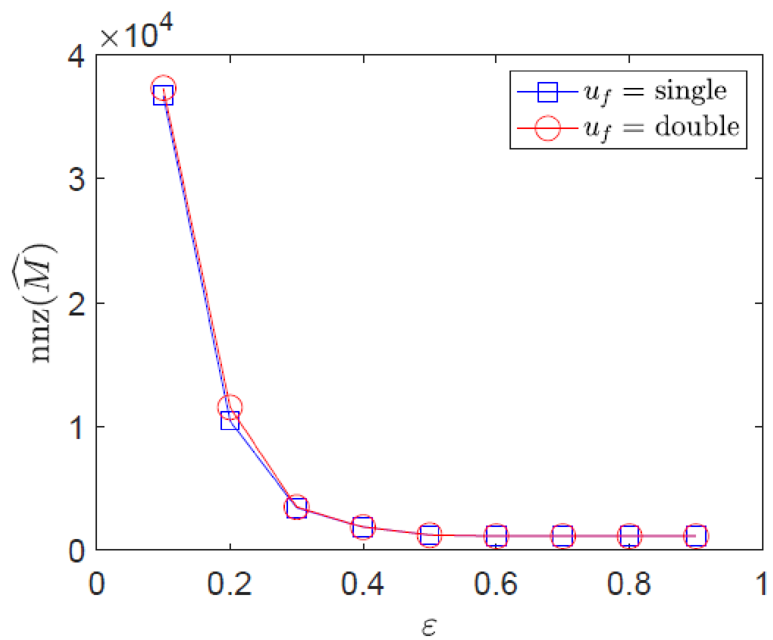


saylr1



A Question

Is there a point in using precision higher than that dictated by $\boldsymbol{u_f}\mathrm{cond}_2(A^T) \le \boldsymbol{\varepsilon}$?

Matrix: bfwa782, $n = 782$, nnz $= 7514$, $\kappa_\infty(A) = 7 \cdot 10^3$, $\mathrm{cond}(A^T) = 1 \cdot 10^3$

$(\boldsymbol{u_f}, \boldsymbol{u}, \boldsymbol{u_r})$ = (**half**, single, double)

Preconditioner	$\kappa_\infty(\tilde{A})$	Precond. nnz	GMRES-IR steps/iteration
SPAI ($\boldsymbol{\varepsilon} = 0.2$)	$2.1e+02$	28053	67 (31, 36)
SPAI ($\boldsymbol{\varepsilon} = 0.5$)	$9.7e+02$	7528	153 (71, 82)

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SPAI ($\varepsilon = 0.2$)	$2.2e + 02$	26801	69 (32, 37)
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Summary and Takeaway

- To efficiently use modern exascale machines, we **need to use mixed precision hardware**

- **Understanding the interaction and balance of errors** from finite precision and sources of algorithmic approximation is thus crucial

- Careful analysis can reveal **not only limitations, but opportunities!**

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Where can you use mixed or low precision?