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A thermodynamically consistent Johnson–Segalman–Giesekus model

numerical simulation of the rod climbing effect

Jakub Cach

joint work with Patrick Farrell, Josef Málek, Karel Tůma

Mathematical Institute of Charles University
Faculty of Mathematics and Physics
Charles University

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Authors and Co-authors



Patrick Farrell¹



Josef Málek²



Karel Tůma²

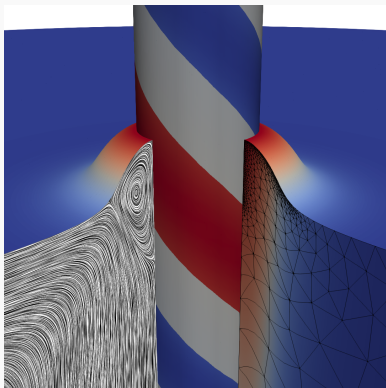
¹ Mathematical Institute, University of Oxford

² Mathematical Institute, Charles University



J. Cach, P. E. Farrell, J. Málek, K. Tůma: A thermodynamically consistent Johnson–Segalman–Giesekus model: numerical simulation of the rod climbing effect, arXiv:2602.01142 (2026).

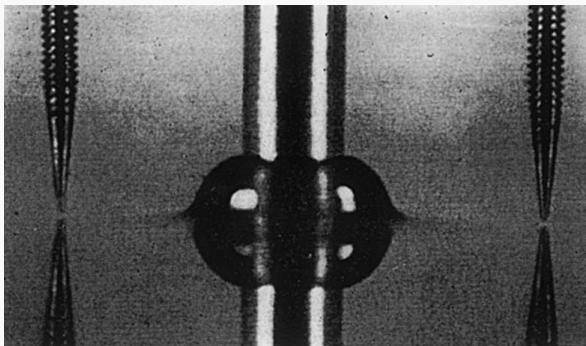
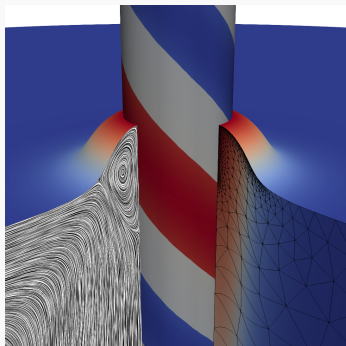
Thermodynamically consistent Johnson–Segalman–Giesekus model



- Rod climbing phenomenon
- Rise of a fluid along a rod rotating about its axis
- Presence of normal stress differences in the model

$$\begin{aligned} \mathbb{T} &= -p\mathbb{I} + 2\mu_s \mathbb{D} + \alpha G(\mathbb{B} - \mathbb{I}) \\ \nabla_a \mathbb{B} + \frac{1}{\tau} \left((1 - \alpha)(\mathbb{B} - \mathbb{I}) + \alpha(\mathbb{B}^2 - \mathbb{B}) \right) &= \mathbb{O} \end{aligned}$$

Rod climbing experiments



G. S. Beavers, D. D. Joseph: The rotating rod viscometer, *Journal of Fluid Mechanics* 69(3), 475–511 (1975). doi:10.1017/S002211207500153X.

Overview

- Two JSG models
- Thermodynamic consistency
- Implementation
- Numerical experiments

Johanson–Segalman–Giesekus models

- Thermodynamically consistent JSG: Model I

$$\begin{aligned}\mathbb{T} &= -p\mathbb{I} + 2\mu_s\mathbb{D} + \alpha G(\mathbb{B} - \mathbb{I}), \\ \nabla_{\sigma} \mathbb{B} + \frac{1}{\tau} \left((1 - \alpha)(\mathbb{B} - \mathbb{I}) + \alpha(\mathbb{B}^2 - \mathbb{B}) \right) &= \mathbb{O}\end{aligned}$$



M. Dostálík, V. Průša, T. Skřivan: On diffusive variants of some classical viscoelastic rate-type models, AIP Conference Proceedings **2107**(1). (2019). doi:10.1063/1.5099919.

- Popular engineering JSG: Model II (later $\mathbb{S} = G(\mathbb{B} - \mathbb{I})$, $\mu_p = \tau G$)

$$\begin{aligned}\mathbb{T} &= -p\mathbb{I} + 2\mu_s \mathbb{D} + \mathbb{S}, \\ \nabla_{\sigma} \mathbb{S} + \frac{1}{\tau} \left(\mathbb{S} + \frac{\alpha}{G} \mathbb{S}^2 \right) &= \frac{2\mu_p}{\tau} \mathbb{D},\end{aligned}$$



M. W. Johnson, D. Segalman: A model for viscoelastic fluid behavior which allows non-affine deformation, Journal of Non-Newtonian Fluid Mechanics (1977).

Johanson–Segalman–Giesekus models

- Thermodynamically consistent JSG: Model I

$$\begin{aligned}\mathbb{T} &= -p\mathbb{I} + 2\mu_s\mathbb{D} + \alpha G(\mathbb{B} - \mathbb{I}), \\ \nabla_{\sigma} \mathbb{B} + \frac{1}{\tau} \left((1 - \alpha)(\mathbb{B} - \mathbb{I}) + \alpha(\mathbb{B}^2 - \mathbb{B}) \right) &= \mathbb{O}\end{aligned}$$



M. Dostálík, V. Průša, T. Skřivan: On diffusive variants of some classical viscoelastic rate-type models, AIP Conference Proceedings **2107**(1). (2019). doi:10.1063/1.5099919.

- Popular engineering JSG: Model II

$$\begin{aligned}\mathbb{T} &= -p\mathbb{I} + 2\mu_s\mathbb{D} + G(\mathbb{B} - \mathbb{I}), \\ \nabla_{\sigma} \mathbb{B} + \frac{1}{\tau} \left((1 - \alpha)(\mathbb{B} - \mathbb{I}) + \alpha(\mathbb{B}^2 - \mathbb{B}) \right) &= 2(1 - \alpha)\mathbb{D}\end{aligned}$$



M. W. Johnson, D. Segalman: A model for viscoelastic fluid behavior which allows non-affine deformation, Journal of Non-Newtonian Fluid Mechanics (1977).

Gordon–Schowalter objective derivatives

- Defined as

$$\overset{\nabla_a}{\dot{\mathbf{A}}} = \dot{\mathbf{A}} - a(\mathbf{D}\mathbf{A} + \mathbf{A}\mathbf{D}) - (\mathbf{W}\mathbf{A} - \mathbf{A}\mathbf{W}), \quad a \in [-1, 1]$$

- Kinematics allow it
- Heuristic microscopic reasoning: slip between polymer chains and velocity field
- In practice: allows to fit experiments
- Rajagopal & Srinivasa framework leads to the upper convected derivative

TD consistency framework

- Reduced thermodynamic identity (isothermal)

$$\xi = \mathbb{T} : \mathbb{D} - \rho \dot{\psi} \quad \text{with} \quad \xi \geq 0$$

- Neo-Hookean Helmholtz free energy ψ per unit mass

$$\psi(\mathbb{B}) = \frac{G}{2\rho} (\text{tr } \mathbb{B} - \ln \det \mathbb{B} - d)$$

$$\rho \dot{\psi} = \frac{G}{2} (\mathbb{I} - \mathbb{B}^{-1}) : \dot{\mathbb{B}}$$

$$\rho \dot{\psi} = \frac{G}{2} (\mathbb{I} - \mathbb{B}^{-1}) : \dot{\mathbb{B}} + \alpha G \mathbb{B} : \mathbb{D}$$

TD consistency results

- Model I:

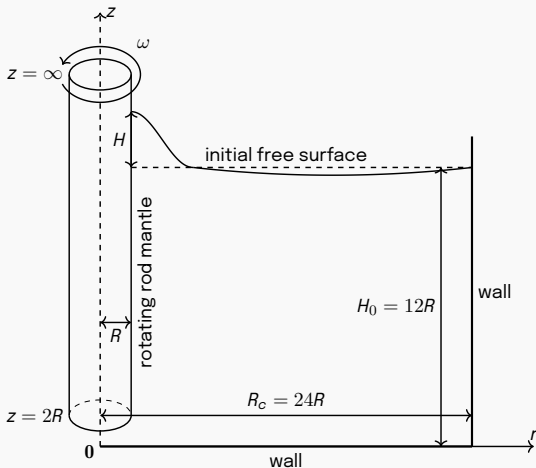
$$\xi = 2\mu_s |\mathbb{D}|^2 + (1 - \alpha) \frac{G}{2\tau} (\mathbb{I} - \mathbb{B}^{-1}) : (\mathbb{B} - \mathbb{I}) + \alpha \frac{G}{2\tau} |\mathbb{B} - \mathbb{I}|^2 \geq 0$$

- Model II:

$$\xi = 2\mu_s |\mathbb{D}|^2 + (1 - \alpha) \frac{G}{2\tau} (\mathbb{I} - \mathbb{B}^{-1}) : (\mathbb{B} - \mathbb{I}) + \alpha \frac{G}{2\tau} |\mathbb{B} - \mathbb{I}|^2 + G(1 - \alpha) (\mathbb{B} + \mathbb{B}^{-1}) : \mathbb{D}$$

Numerical simulations ingredients

- Rod-climbing
- ALE method
- Geometry & fluid properties
- Firedrake
- Results



- Governing equations

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega,$$

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbb{T} + \mathbf{f} \quad \text{in } \Omega,$$

$$\mathbb{T} = -p\mathbb{I} + 2\mu_s\mathbb{D} + \alpha G(\mathbb{B} - \mathbb{I}),$$

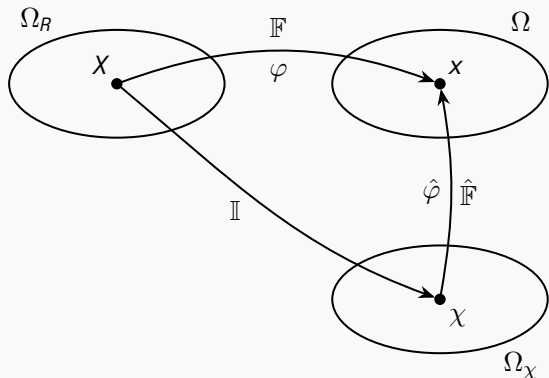
$$\nabla_{\alpha} \mathbb{B} + \frac{1}{\tau} \left((1 - \alpha)(\mathbb{B} - \mathbb{I}) + \alpha(\mathbb{B}^2 - \mathbb{B}) \right) = \mathbb{O} \quad \text{in } \Omega$$

- Supplied by free-surface boundary condition

$$-\Delta_{\chi} \hat{\mathbf{u}} = 0 \quad \text{in } \Omega_{\chi},$$

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} = \mathbf{v} \quad \text{on } \partial\Omega_{\chi}$$

ALE mapping



- Mesh configuration (χ, Ω_χ)
- New variable deformation $\hat{\mathbf{u}}$
- Mapping $\hat{\varphi}$ from Ω_χ to Ω as $\hat{\varphi} : \chi \longrightarrow \mathbf{x} := \chi + \hat{\mathbf{u}}$
- Transformations

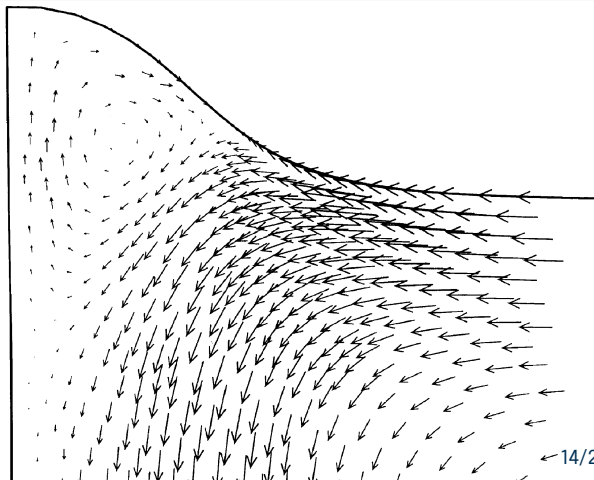
$$\hat{\mathbb{F}} := \frac{\partial \varphi_t}{\partial \chi} = \mathbb{I} + \nabla_\chi \mathbf{u}_t,$$

$$\nabla_{\mathbf{x}} \mathbf{v}_t = (\nabla_\chi \mathbf{v}_t) \hat{\mathbb{F}}^{-1},$$

$$\dot{\alpha} = \frac{\partial \alpha}{\partial t} \Big|_\chi + \left[\hat{\mathbb{F}}^{-1} \left(\mathbf{v}_t - \frac{\partial \mathbf{u}_t}{\partial t} \right) \right] \cdot \nabla_\chi \alpha$$

Steady state in ALE configuration

- Axi-symmetry: $\hat{u}_\varphi = 0$
- $\hat{\mathbf{u}} = (\hat{u}_r, 0, \hat{u}_z)$
- Free-surface kinematic condition
 $\partial_t h = v_z - v_r \partial_r h$
- Setting $h := z + \hat{u}_z$
- Reduces $\hat{\mathbf{u}} = (0, 0, \hat{u}_z)$



System to solve

$$\begin{aligned}
 & \int_{\Omega_\chi} r \hat{J} \operatorname{tr} \left((\nabla_{\hat{\chi}} \mathbf{v}) \hat{\mathbb{F}}^{-1} \right) \psi = 0, \\
 & \operatorname{Re} \int_{\Omega_\chi} r \hat{J} \left[\frac{\partial \mathbf{v}}{\partial t} + (\nabla_{\hat{\chi}} \mathbf{v}) \left(\hat{\mathbb{F}}^{-1} \left(\mathbf{v} - \frac{\partial \hat{\mathbf{u}}}{\partial t} \right) \right) \right] \cdot \phi + \int_{\Omega_\chi} r \hat{J} \hat{\mathbb{T}} \hat{\mathbb{F}}^{-T} \cdot \nabla_{\hat{\chi}} \phi \\
 & + \frac{1}{Ca} \int_{\Gamma_\chi^{\text{free}}} r \|\hat{\mathbb{F}}^{-T} \mathbf{n}\| \left[\mathbb{I} - \frac{\hat{\mathbb{F}}^{-T} \mathbf{n}}{\|\hat{\mathbb{F}}^{-T} \mathbf{n}\|} \otimes \frac{\hat{\mathbb{F}}^{-T} \mathbf{n}}{\|\hat{\mathbb{F}}^{-T} \mathbf{n}\|} \right] (\nabla_{\hat{\chi}} \phi) \hat{\mathbb{F}}^{-1} + \frac{1}{St} \int_{\Omega_\chi} r \hat{J} \mathbf{e}_z \cdot \phi = 0, \\
 & \hat{\mathbb{T}} = -p \mathbb{I} + 2 \frac{\mu_s}{\mu_s + \mu_p} \mathbb{D}_{\hat{\chi}} + \frac{a}{We} \frac{\mu_p}{\mu_s + \mu_p} (\mathbb{B} - \mathbb{I}), \\
 & \int_{\Omega_\chi} r \hat{J} \left. \nabla_a \mathbb{B} \right|_{\hat{\chi}} \cdot \mathbb{A} + r \hat{J} \left[\frac{1-\alpha}{We} (\mathbb{B} - \mathbb{I}) + \frac{\alpha}{We} (\mathbb{B}^2 - \mathbb{B}) \right] \cdot \mathbb{A} = 0, \\
 & \int_{\Omega_\chi} r \partial_z \hat{u}_z \partial_z \hat{w}_z - \int_{\partial \Omega_\chi} r (\partial_z \hat{u}_z) \hat{w}_z - \int_{\partial \Omega_\chi} r \left[\frac{\partial \hat{u}_z}{\partial t} + (v_r \partial_r \hat{u}_z - v_z) \right] \left(\partial_z \hat{w}_z - \frac{\beta_{st}}{h_{\min}} \hat{w}_z \right) = 0,
 \end{aligned}$$

Implementation

- Finite element method using the Firedrake library
- High-order discretization ($p = 4$): Scott-Vogelius pair $([CG_p]^3 - DG_{p-1})$
- Matching high-order spaces: $[CG_{p-1}]_{\text{sym}}^{3 \times 3}$ for extra stress, CG_p for mesh displacement
- High-order geometry of free surface preserved via virtual-ALE formulation
- Second-order three-step θ -scheme (Glowinski) with excellent volume preservation
- Fully monolithic coupling (no splitting, no additional stabilization required)
- Efficient nonlinear solution: lagged-Jacobian strategy + LU reuse
- High accuracy at moderate cost: $\sim 130\text{k}$ DoF, parallel, minutes on a laptop
- No interface reconstruction or CFL restrictions

Results

- Self-contained mesh-convergence test
- Comparison with numerical solutions



R. A. Figueiredo, C. M. Oishi, A. M. Afonso, I. V. M. Tasso, J. A. Cuminato: A two-phase solver for complex fluids: Studies of the Weissenberg effect, International Journal of Multiphase Flow (2016).



X.-L. Luo: Numerical simulation of Weissenberg phenomena—the rod-climbing of viscoelastic fluids, Computer Methods in Applied Mechanics and Engineering. (1999)

- Comparison with experiments



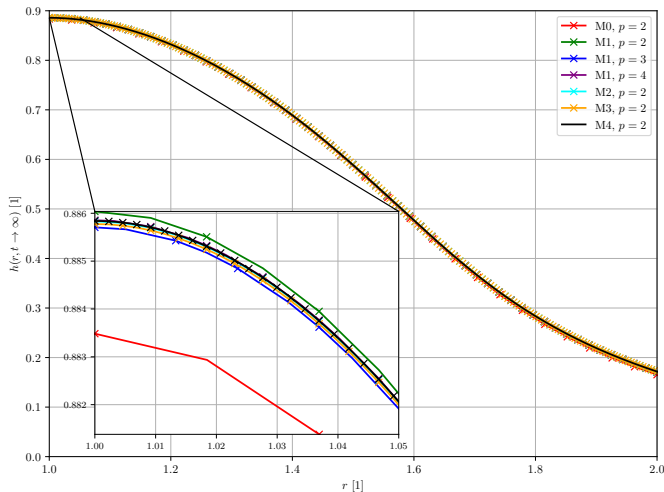
G. S. Beavers, D. D. Joseph: The rotating rod viscometer, Journal of Fluid Mechanics. (1975).

- Comparison of Model I and Model II
- Comparison with analytical solution (Giesekus model)

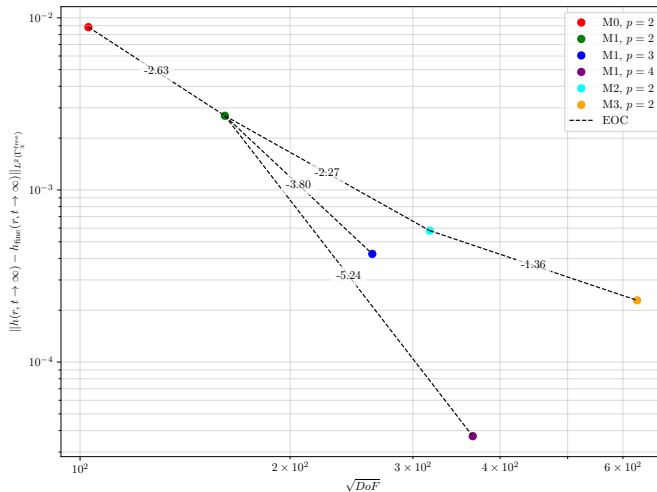


T. Ruangkriengsin, R. Brandão, K. Wu, J. Hwang, E. Boyko, H. A. Stone: Transient rod climbing in a viscoelastic fluid, Journal of Fluid Mechanics (2025).

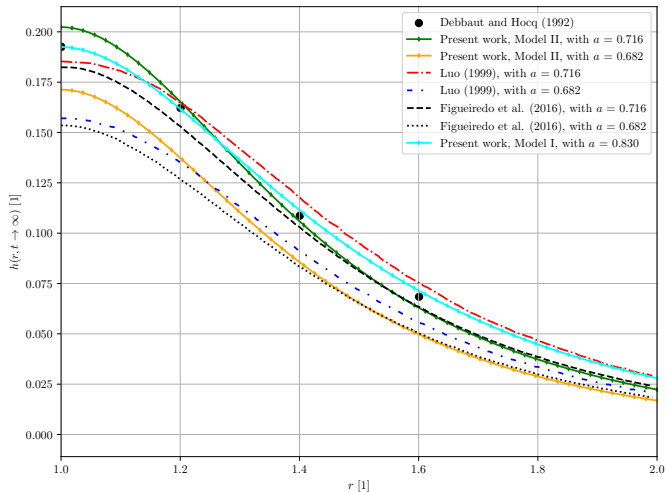
Mesh-convergence test (Oldroyd-B)



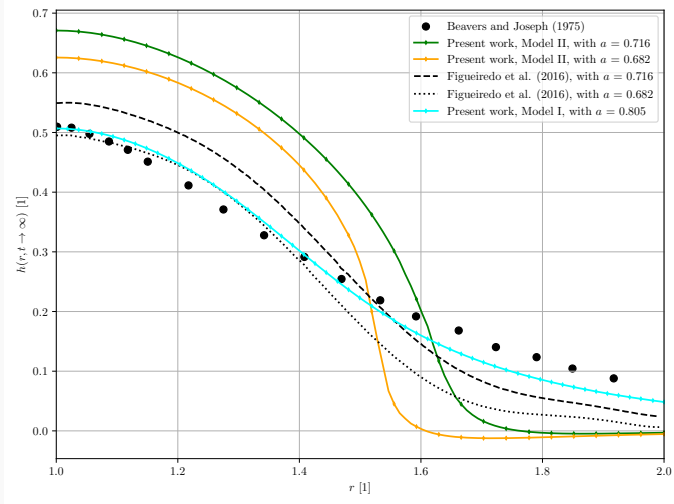
Mesh-convergence test II



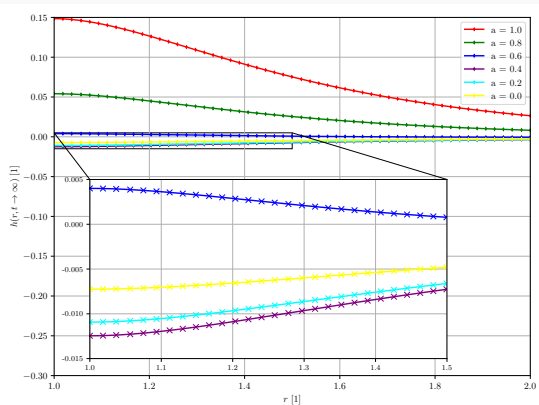
Comparison at slow rotations (JS)



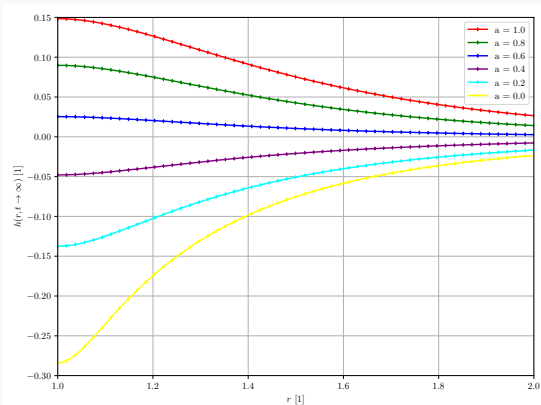
Comparison at high rotations (JS)



Comparison of Models I and II: $\alpha \in [0, 1]$

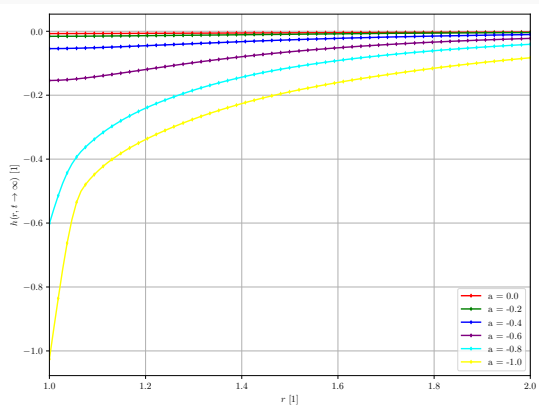


(a) Model I ($\alpha = 0$)

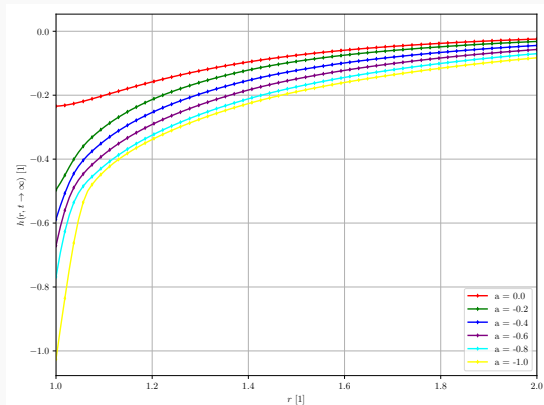


(b) Model II ($\alpha = 0$)

Comparison of Models I and II: $\alpha \in [-1, 0]$



(a) Model I ($\alpha = 0$)

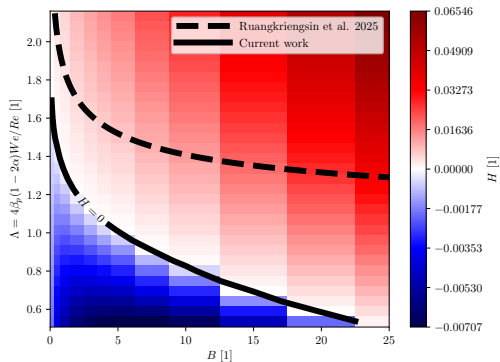


(b) Model II ($\alpha = 0$)

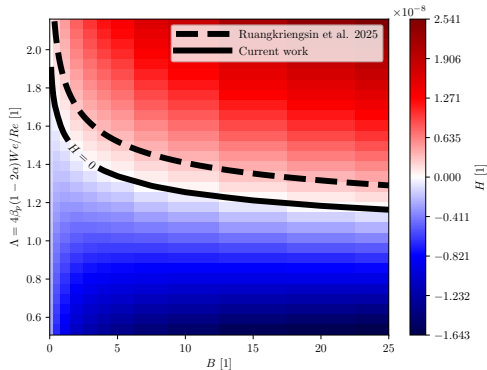
Comparison with analytical solution ($a = 1$)



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(a) $We = 0.195, G = 0.069$



(b) $We = 9.73 \times 10^{-3} \ll 1, G = 138 \gg 1$

Summary

- ✓ Model I satisfies the 2nd law of thermodynamics
- ✓ Model I exhibits rod-climbing for all $\alpha \in [0, 1]$
- ✓ Model I shows superior agreement with experiments
- ✗ Model II violates the 2nd law of thermodynamics
- ✓ Efficient and accurate code

References and acknowledgement



1) **J. Cach, P. E. Farrell, J. Málek, K. Tůma**: A thermodynamically consistent Johnson–Segalman–Giesekus model: numerical simulation of the rod climbing effect, arXiv:2602.01142 (2026).



2) **David A. Ham et al.** *Firedrake User Manual*. First edition. Imperial College London and University of Oxford and Baylor University and University of Washington. May 2023. DOI: 10.25561/104839.

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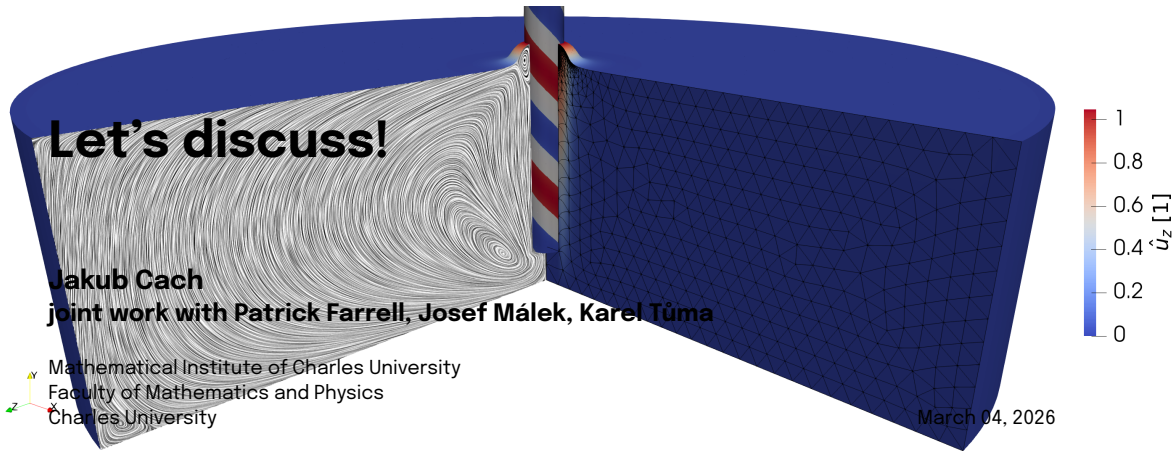
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Let's discuss!

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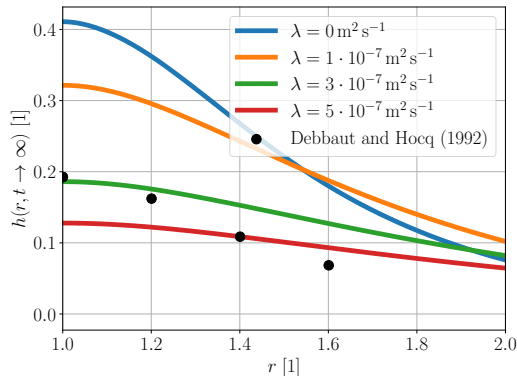
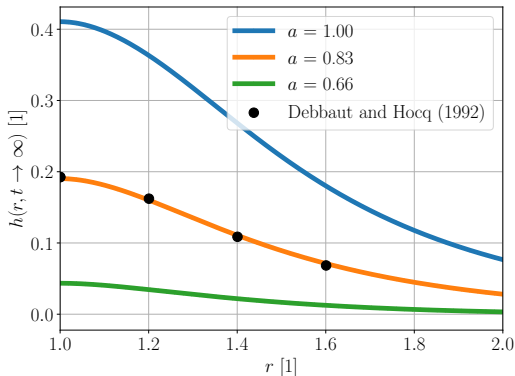
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Outlook

- Stress diffusion $\lambda \Delta \mathbb{B}$



- Best fit $a \approx 0.8$, $\lambda \approx 10^{-8} \text{ m}^2 \text{ s}^{-1}$
- Conjecture: Every viscoelastic model should come with a stress diffusion