A Piecewise Conical Stenosis Model for Predicting Pressure Losses (FFR)

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Introduction

We present an analytical framework for predicting steady pressure losses in straight, axisymmetric tubes with arbitrarily varying diameter at small to moderately high Reynolds numbers. Classical reduced-order models either rely on lumped descriptions or on lubrication-type approximations, limiting their accuracy for geometries with finite taper angles. To overcome these limitations, we formulate a segment-wise model based on conical elements, for which a reduced ordinary differential equation description is derived and solved analytically or semi-analytically. The proposed approach can be interpreted as a discrete counterpart of lubrication theory that remains valid for finite geometric gradients. Arbitrary radius profiles are treated by decomposing the geometry into conical segments and assembling the corresponding local solutions, in particular, to evaluate the fractional flow reserve (FFR).

Starting point

- Steady incompressible Navier-Stokes equations:

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega, \\ (\mathbf{v}\cdot\nabla)\mathbf{v} &= -\nabla p + \frac{1}{\mathrm{Re}}\Delta \mathbf{v} && \text{in } \Omega, \\ \mathbf{v} &= 0 && \text{on } \Gamma_{\text{wall}} \subset \partial\Omega, \\ \int_{\Gamma_{\text{inflow}}} \mathbf{v}\cdot\mathbf{n} &= Q && \text{on } \Gamma_{\text{inflow}} \subset \partial\Omega \end{aligned} \qquad (1)$$

  are used both for the reduction and for numerical simulations
- Reynolds nb. in radius-varying vessels $\mathrm{Re}(R) := \frac{Q}{\pi\nu R}$ is **local** quantity
- Full 3D simulations are costly; reduced models offer efficiency
- Goal: capture essential physics with minimal computational effort

Model reduction

- Axisymmetric flow in spherical coordinates (r, θ, ϕ)
- Assuming dominance of the radial flow: $\mathbf{v}(r,\theta) = (v_r(r,\theta), 0, 0)$
- Continuity equation leads to: $v_r(r,\theta) = \frac{1}{r^2}F(\theta)$
- Momentum equation leads to $p(r,\theta) = \frac{2}{\mathrm{Re}\, r^3}F(\theta) + G(r)$ and

$$F''(\theta) + \cot\theta\, F'(\theta) + 6F(\theta) + \frac{2\mathrm{Re}}{r}F^2(\theta) = \mathrm{Re}\, r^4\, G'(r), \ (r,\theta) \in \Omega^{2D},$$
$$F(\alpha) = 0, \ F'(0) = 0,$$
$$Q = 2\pi \int_0^\alpha F(\theta)\sin\theta\, d\theta$$

- The convective term $(F^2(\theta))$ prevents a clean separation of variables

Approximation and solve

- Quasi-linear approximation uses the Stokes profile:

$$F''(\theta) + \cot\theta\, F'(\theta) + 6F(\theta) = \mathrm{Re}\, r^4\, G'(r) - \frac{2\mathrm{Re}}{r}\overline{F^2(\theta)}$$

  and leads to a pressure formula with an explicit $f(\alpha)$

$$\overline{p}_{\text{quasi}}(r) = \frac{Q\mu}{\pi r^3}\frac{\cos(\alpha)(\cos(\alpha)+1)}{(2\cos(\alpha)+1)(1-\cos(\alpha))^2} + \frac{\rho Q^2}{r^4}f(\alpha) \qquad (2)$$

- An effective nonlinear approximation around a prescribed r^*_{given}:

$$F''(\theta) + \cot(\theta)F'(\theta) + 6F(\theta) + \frac{2Re}{r^*_{\text{given}}}F^2(\theta) = C, \qquad (3)$$
$$(r^*)^4 G'(r^*) = C$$

Numerical methods & Implementation

Direct numerical simulation (DNS) of PDE (1)
- Firedrake finite element implementation in cylindrical coordinates
- Mesh generation: Netgen (with curved boundaries)
- Spatial discretization: adapted Scott–Vogelius elements (CG4–DG3)
- Pressure drop: average over inflow and outflow boundaries
- Continuation over wide ranges of Reynolds numbers

Numerical solution of ODE (3)
- Firedrake formulation with a Lagrange multiplier

Configurations (represents 3D volume by revelation around lower symm. wall)

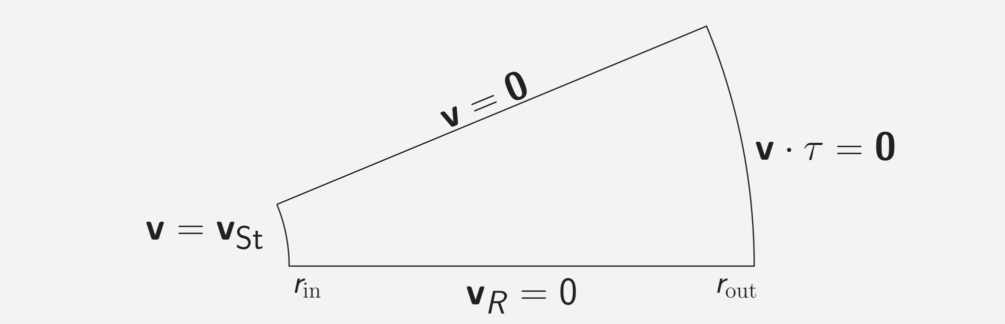


Figure: Cut cone (wedge) domain: inner radius $r_{in} = R_{in}/\sin\alpha$, outer radius $r_{out} = r_{in} + L$, opening angle α.

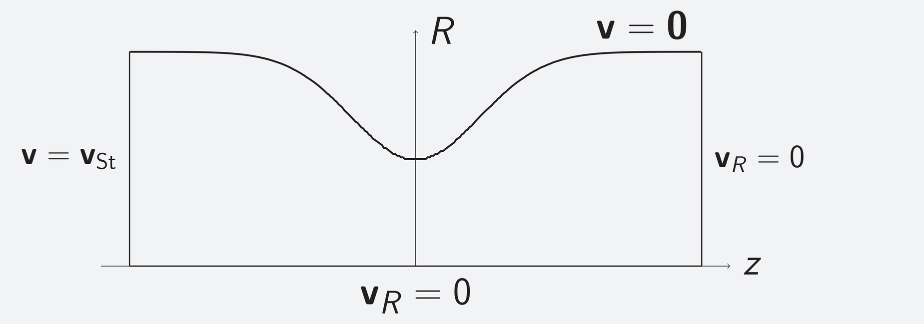
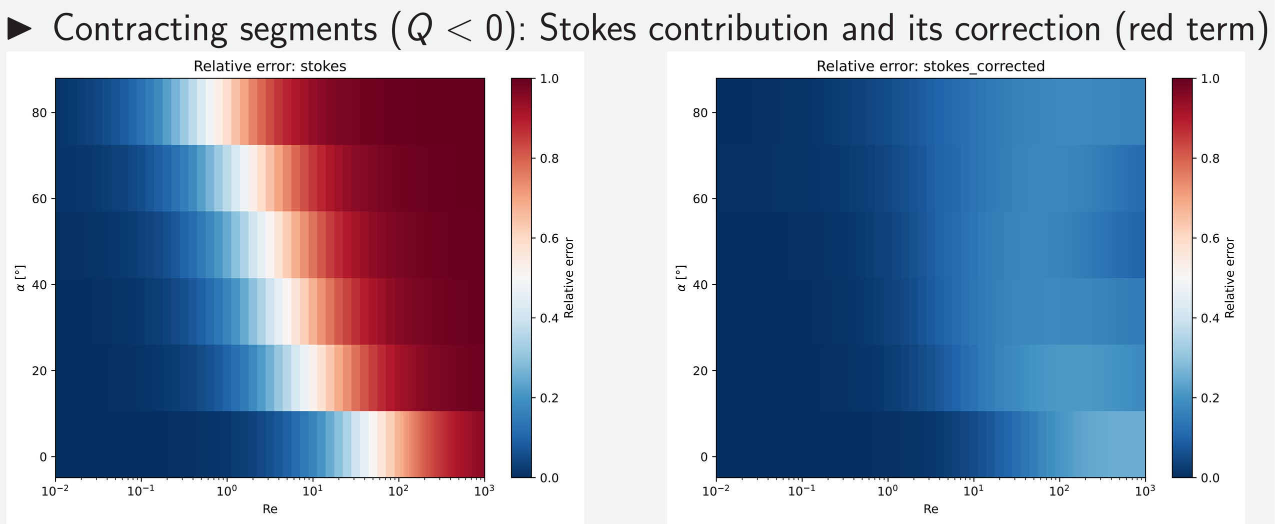


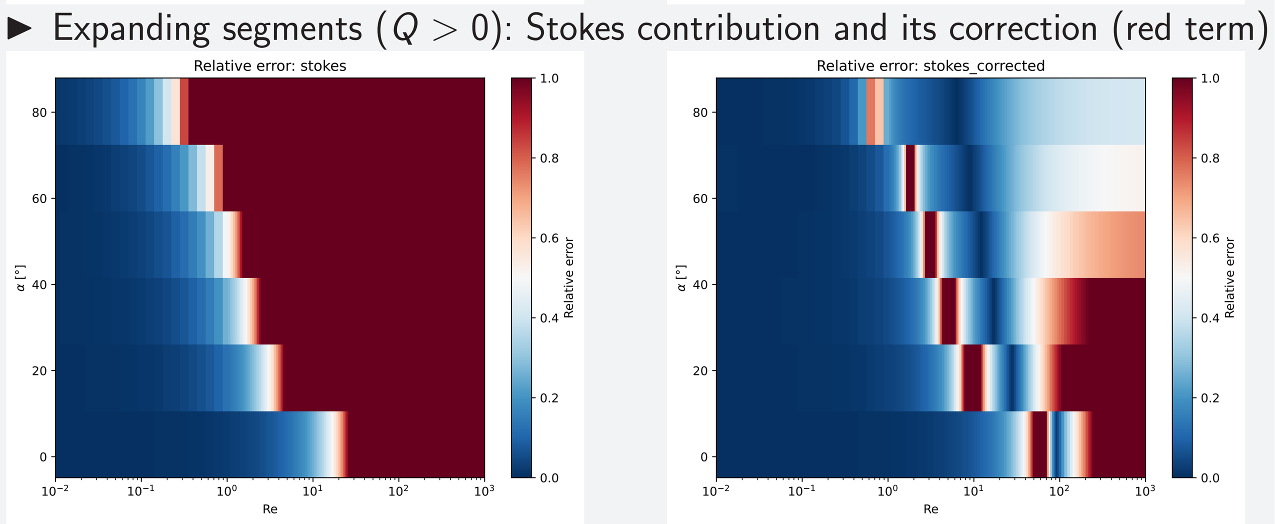
Figure: Study case: abruptly stenosed vessel domain: $R(z) = R_{in} - \frac{1}{2}R_{in}e^{-1000z^2}$ on $z \in [-10\ \mathrm{cm}, 10\ \mathrm{cm}]$, $R_{in} = 3$ mm.

Conical segments: relative error of formula (2) vs. DNS (1)

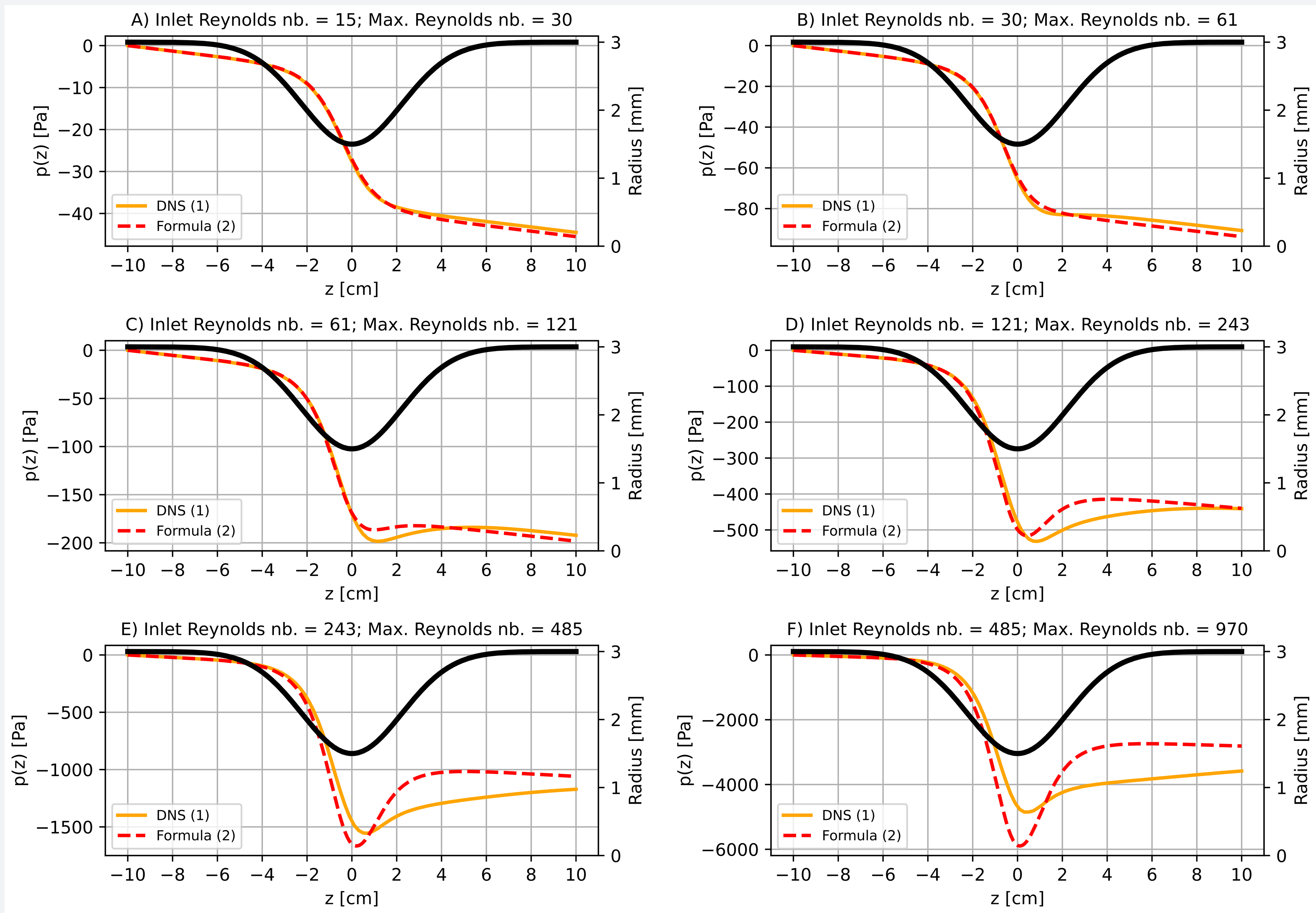
- Contracting segments ($Q < 0$): Stokes contribution and its correction (red term)



- Expanding segments ($Q > 0$): Stokes contribution and its correction (red term)

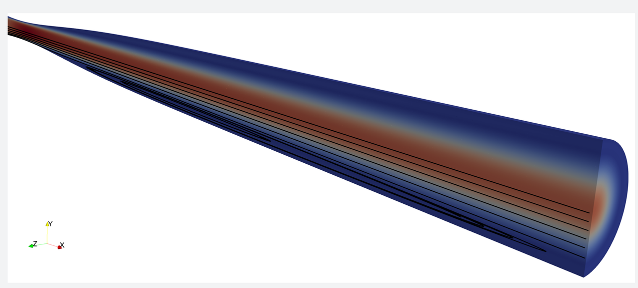


Study case: pressure drop for characteristic human-vessel parameters, with flow-rate (Re) varied



Study case: D) parameters & F) DNS (1) velocity field

case D)
- $\mu = 3.896\ \mathrm{mPa\cdot s}$
- $\rho = 1050\ \mathrm{kg\cdot m^{-3}}$
- $Q = 4.24\ \mathrm{mL\cdot s^{-1}}$
- $\mathrm{Re}(R) := \frac{\rho Q}{\pi\mu R}$

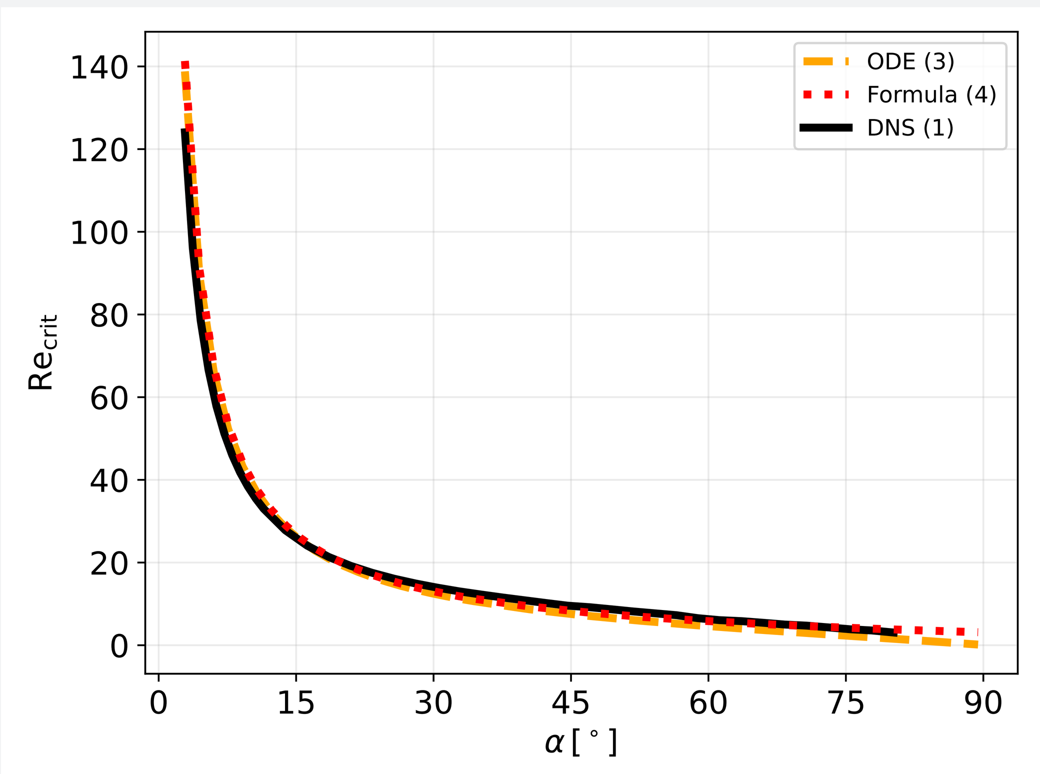


Expanding conical segments: onset of separation layer

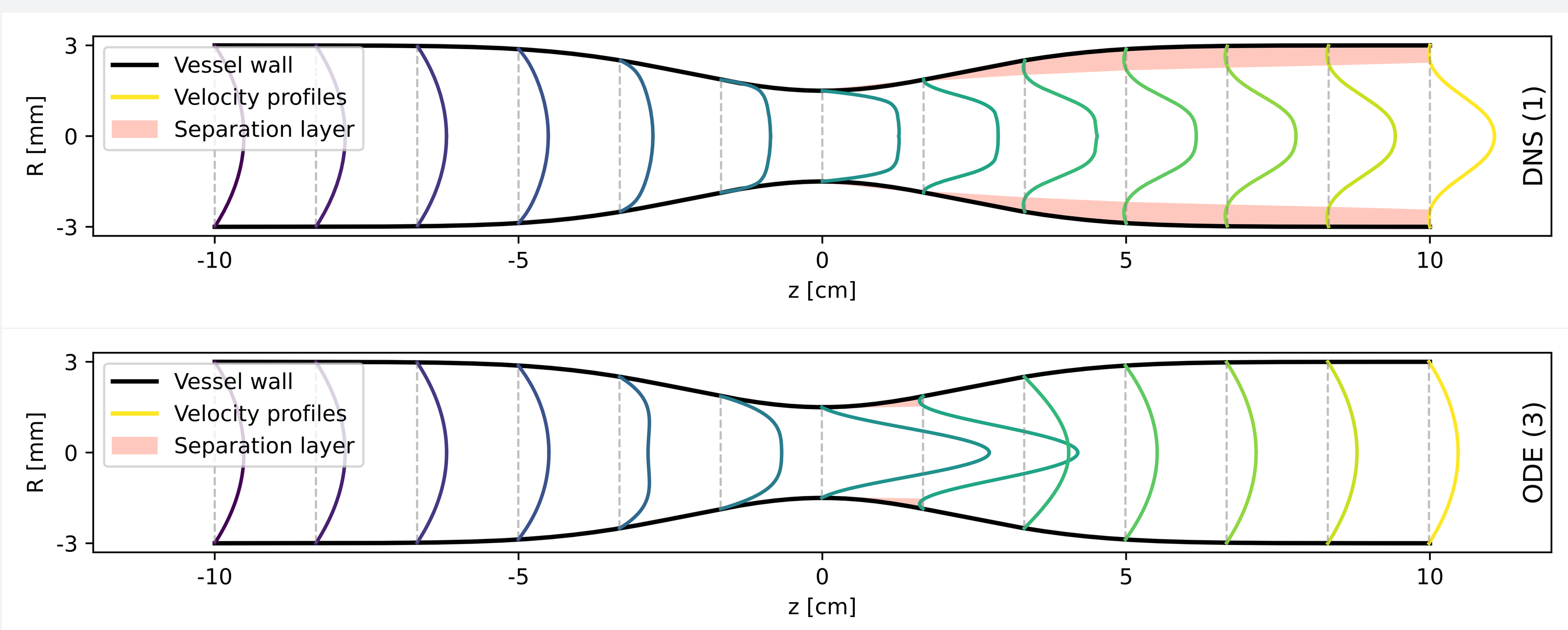
- Onset of backflow $F'(\theta) \geq 0$ from a first-order perturbation expansion of (3):

$$\mathrm{Re}_{crit}(\alpha) = \frac{9\pi}{4\alpha}\left(1 - \frac{\alpha^2}{8}\right) \qquad (4)$$

- We compare formula (4) with a numerical solution to ODE (3) and PDE (1):



Velocity profiles in study case F): DNS (1) vs. piecewise approx. by semi-analytical solutions of (3)



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