

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University



MINISTRY OF EDUCATION,
YOUTH AND SPORTS

On bifurcations and traction forces on an obstacle in incompressible flow

Jakub Cach
joint work with Jan Blechta, Sebastian Schwarzacher, Karel Tůma

Mathematical Institute of Charles University
Faculty of Mathematics and Physics
Charles University

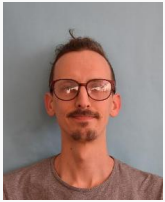
May 11, 2026

Authors and Co-authors

FACULTY OF MATHEMATICS AND PHYSICS
Charles University



Karel Tůma[1]



Jan Blechta[1]



Sebastian Schwarzacher[1,2]

[1] Mathematical Institute, Charles University
[2] Mathematical Institute, Uppsala University



Jakub Cach, Karel Tůma, Jan Blechta, & Sebastian Schwarzacher, *On bifurcations and traction forces on an obstacle in incompressible flow*, **arXiv:2512.15424 [physics.flu-dyn] (2025)**

Equations and traction

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

- Unsteady incompressible Navier-Stokes equations

$$\begin{aligned}
\rho(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) &= \operatorname{div}\mathbb{T} \quad \text{in } \Omega,\\
\operatorname{div}\mathbf{v} &= 0 \quad \text{in } \Omega,\\
\mathbb{T} &= -p\mathbb{I} + \mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T),\\
\mathbf{v} &= \mathbf{v}_D^i \quad \text{on } \Gamma_i \subset \partial\Omega,\\
\mathbb{T}\mathbf{n} &= 0 \quad \text{on } \partial\Omega\backslash\cup\Gamma_i.
\end{aligned}$$

- Traction

$$\mathbf{t}(\theta;\mathrm{Re}) := \mathbb{T}\mathbf{n}.$$

- Discretization: Firedrake; Taylor-Hood CG(2,1); monolitic; Newton+LU; Crank-Nicolson

Schäfer–Turek benchmark

- $t_{\mathsf{drag}}(\theta; \mathrm{Re}) := t_x(\theta; \mathrm{Re})$
- $t_{\mathsf{lift}}(\theta; \mathrm{Re}) := t_y(\theta; \mathrm{Re})$

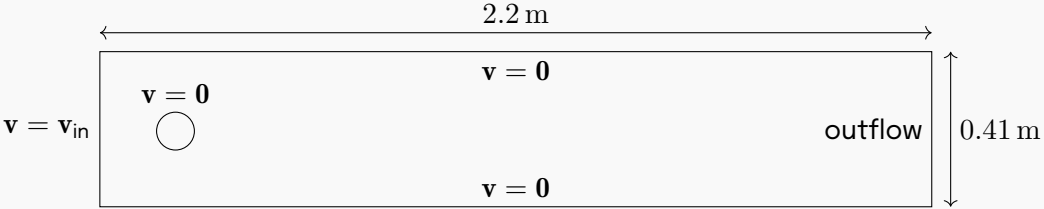


Figure: The Schäfer–Turek benchmark. The lower left corner is at $(0\,\mathrm{m}, 0\,\mathrm{m})$ and the cylinder center is located at $(0.2\,\mathrm{m}, 0.2\,\mathrm{m})$ with radius $R = 0.05\,\mathrm{m}$. The domain is slightly asymmetric. The benchmark does not require use of a specific outflow condition.

Aim

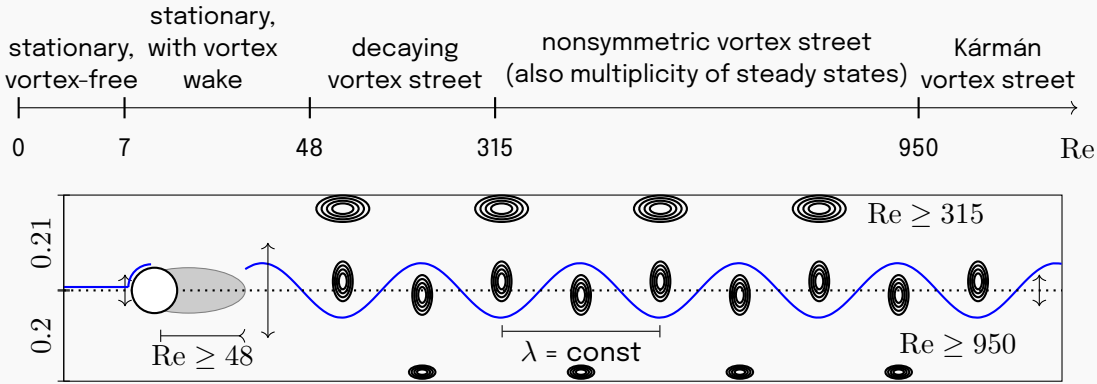
FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

- Explain why do we study traction?
- Systematically observe all transition in the flow up to $\mathrm{Re} = 1000$
- A posteriori connect critical points in the traction $\partial_{\mathrm{Re}}\mathbf{t} = 0$ to transitions
- Explain the empirical relations in framework of bifurcation theory
- Physical interpretation of the results
- Classification of bifurcation in our framework

Open question:

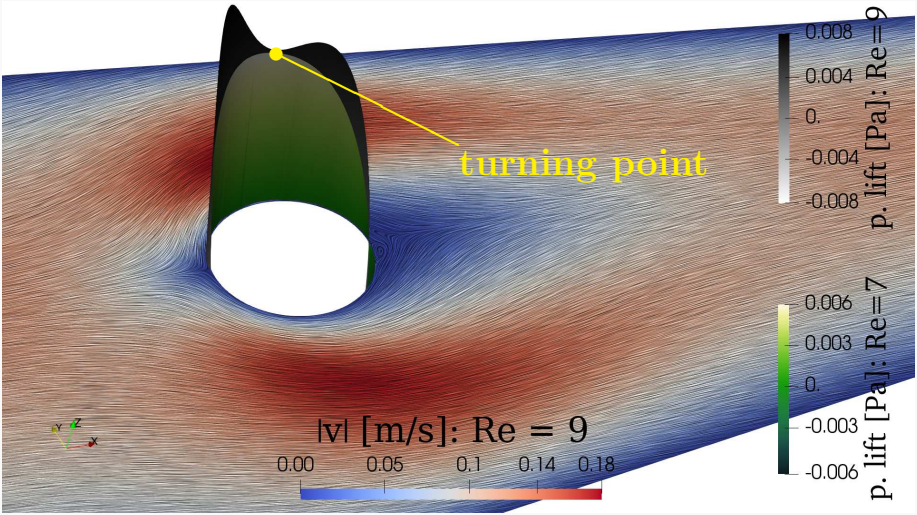
- unique time-periodic global-in-time attractor

Pointwise traction – fluid bifurcation



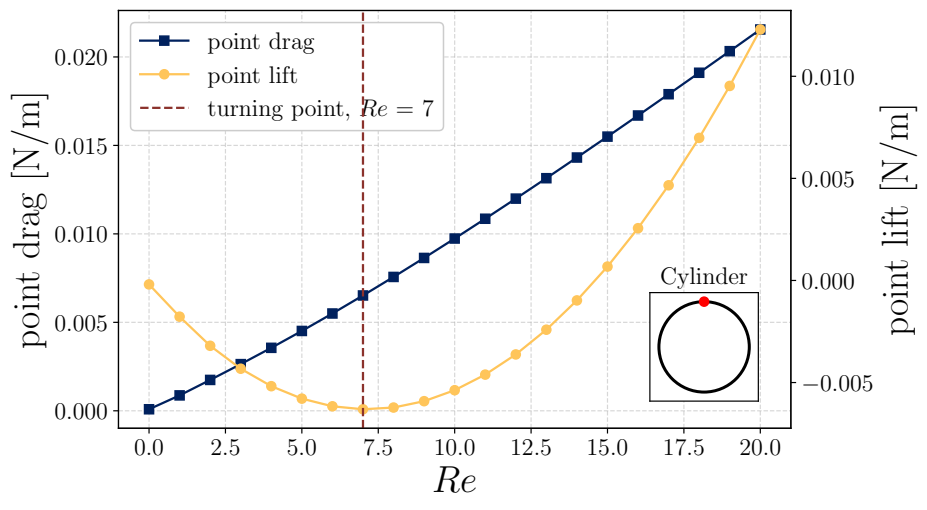
Pointwise traction

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University



Pointwise traction – turning point

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University



Navier-Stokes theory – overview

FACULTY OF MATHEMATICS AND PHYSICS
Charles University

- steady BVP non-unique in 2D, 3D (except low Re)
- IBVP unique in 2D (Ladyženskaja)
- uniqueness of time-periodic solutions in 2D, 3D
  - steady solution only unique time-per. solution in case of a single obstacle flow (low Re)
- questions:
  - what makes stable solution?
  - are long-time attractors unique?
  - are eigenvalues the key?

How we numerically analyze

FACULTY OF MATHEMATICS AND PHYSICS
Charles University

- global velocity field of the unsteady solution (long-time attractor)
- global velocity field of the steady solution
- traction profile of the steady solution
- eigenvalues of the Linear Perturbation Theory (LPT) operator (2D and 3D variant)
- deflated continuation of steady equations

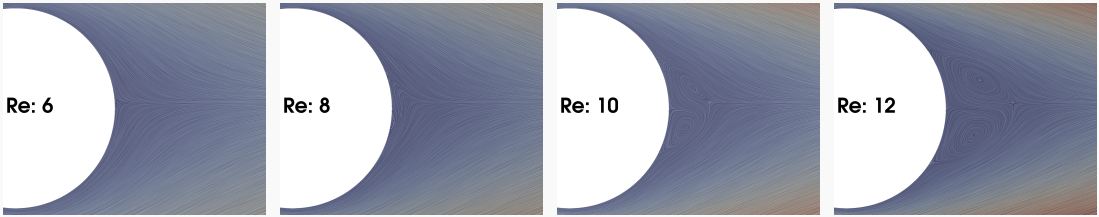
LPT – equations

Expanding the solution around the steady state $\mathbf{v}_{\text{st}}$ as $\mathbf{v} = \mathbf{v}_{\text{st}} + \varepsilon\tilde{\mathbf{v}}$ and assuming $\tilde{\mathbf{v}} = e^{\lambda t}\hat{\mathbf{v}}$, we neglect terms $O(\varepsilon^2)$ and obtain:

$$
\begin{aligned}
\operatorname{div}\hat{\mathbf{v}} &= 0 \quad \text{in } \Omega,\\
\lambda\hat{\mathbf{v}} &= \operatorname{div}\hat{\mathbb{T}} - \hat{\mathbf{v}}\cdot\nabla\mathbf{v}_{\text{st}} - \mathbf{v}_{\text{st}}\cdot\nabla\hat{\mathbf{v}} \quad \text{in } \Omega,\\
\hat{\mathbb{T}} &= -\hat{p}\mathbb{I} + \mu(\nabla\hat{\mathbf{v}} + \nabla\hat{\mathbf{v}}^T),\\
\hat{\mathbf{v}} &= 0 \quad \text{on } \Gamma_i \subset \partial\Omega,\\
\hat{\mathbb{T}}\mathbf{n} &= 0 \quad \text{on } \partial\Omega \setminus \cup_i \Gamma_i.
\end{aligned}
\tag{1}
$$

Results: Low Re flow ($\mathrm{Re} < 7$)

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

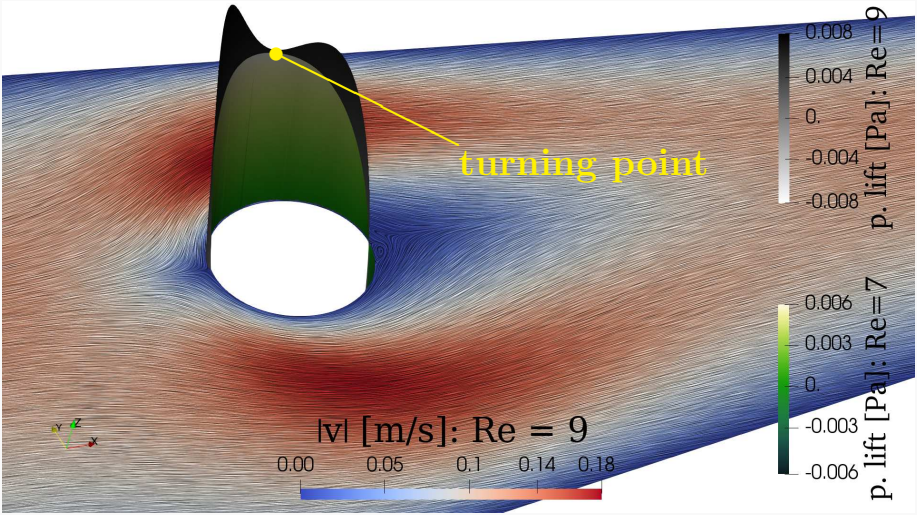


(a) $\mathrm{Re} = 6$ (b) $\mathrm{Re} = 8$ (c) $\mathrm{Re} = 10$ (d) $\mathrm{Re} = 12$

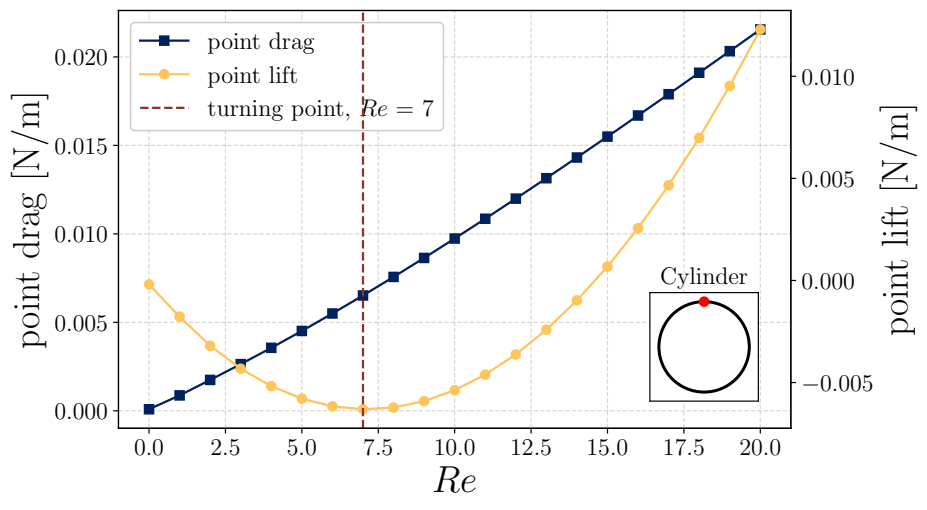
Figure: Velocity field on the downstream side of the cylinder in line integral convolution plot (streamlines). A pair of stationary vortices appears at $\mathrm{Re} = 7$ and grows in size with increasing Re.

Pointwise traction

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University



Pointwise traction – turning point



Turning point – global view

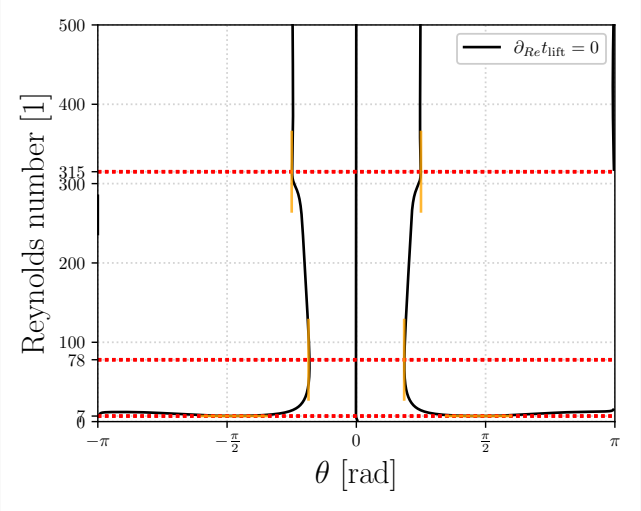


Figure: Pointwise lift: $\partial_{\mathrm{Re}} \boldsymbol{t}_{\mathrm{lift}}(\theta, \mathrm{Re}) = 0$ shown in black

Non-trivial long-time solution ($\mathrm{Re} \geq 48$)

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

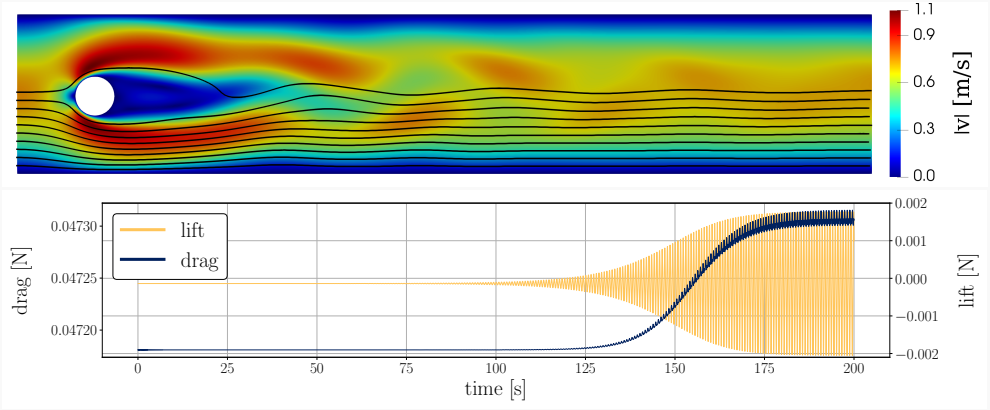
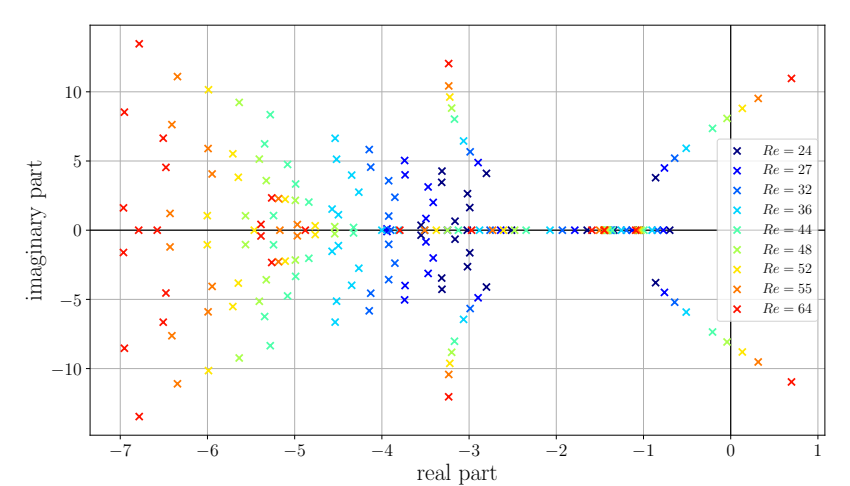


Figure: Numerical simulation of evolution problem at $\mathrm{Re} = 50$.

Hopf bifurcation (Re = 48)

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University



LPT eigemodes viewpoint

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

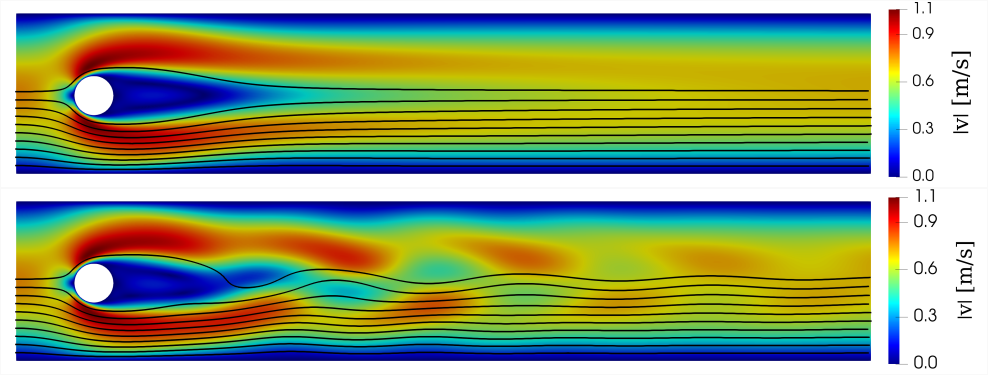
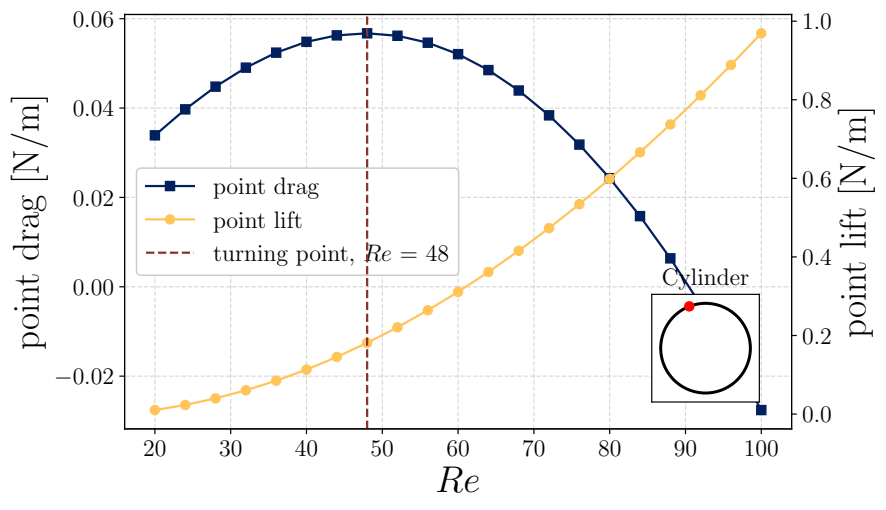


Figure: Eigenmode reconstruction for $\mathrm{Re} = 50$.

Traction – turning point (Re = 48)

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University



Turning point – global view

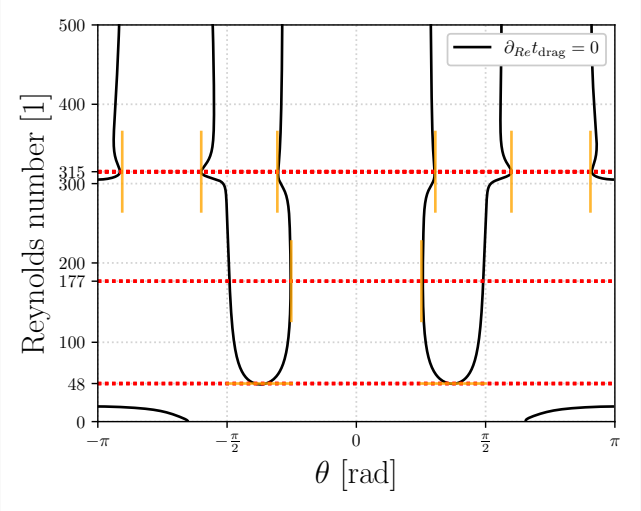
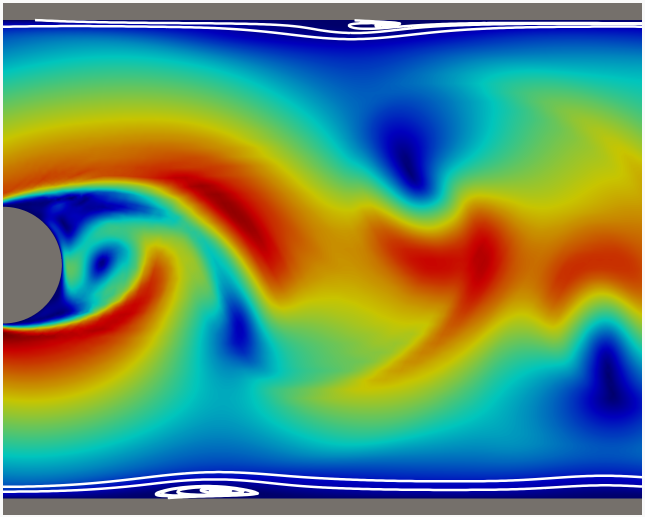


Figure: Pointwise drag: $\partial_{\mathrm{Re}} t_{\mathrm{drag}}(\theta, \mathrm{Re}) = 0$ shown in black

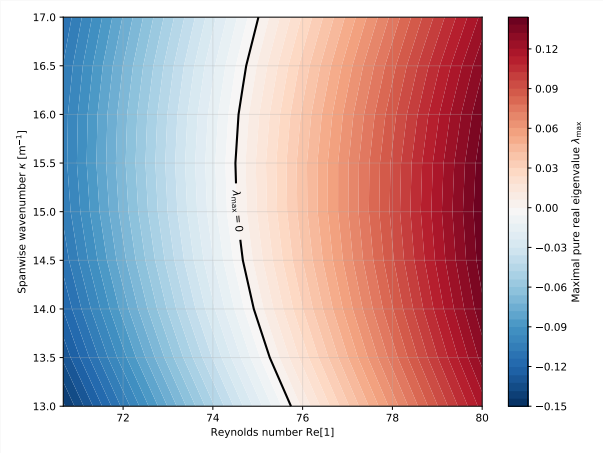
Wall effect: Vortices along boundaries ($\mathrm{Re} = 78$)

FACULTY OF MATHEMATICS AND PHYSICS
Charles University



Steady 3D bifurcation: mode E ($\mathrm{Re} = 75$)

FACULTY OF MATHEMATICS AND PHYSICS
Charles University



- LPT as 3D perturbation of 2D base flow
- spatial wavelenght κ in spanwise coord. z must be solver for
- observation: spatial period $L_z \approx$ domain width

Turning point – global view

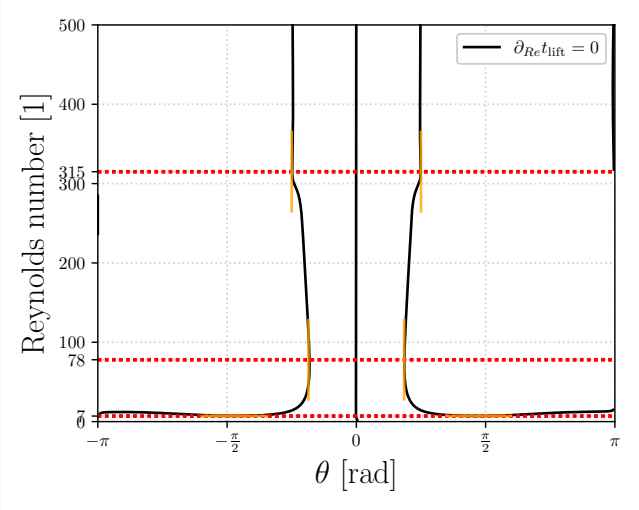


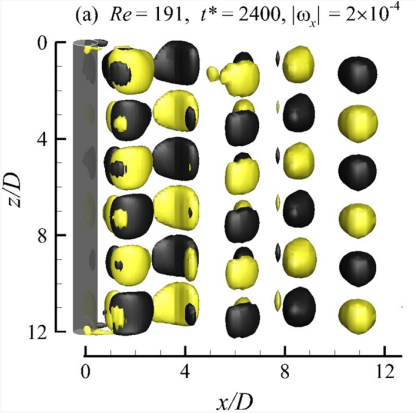
Figure: Pointwise lift: $\partial_{\mathrm{Re}} \boldsymbol{t}_{\mathrm{lift}}(\theta, \mathrm{Re}) = 0$ shown in black

Animation $\mathrm{Re} = 100$

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Unsteady 3D bifurcation: mode A $\mathrm{Re} \approx 177$

FACULTY OF MATHEMATICS AND PHYSICS
Charles University



H. Jiang, L. Cheng, F. Tong, S. Draper, and H. An, *Stable state of Mode A for flow past a circular cylinder*, Physics of Fluids, vol. 28, no. 10, 2016. doi: 10.1063/1.4964379.

Turning point – global view

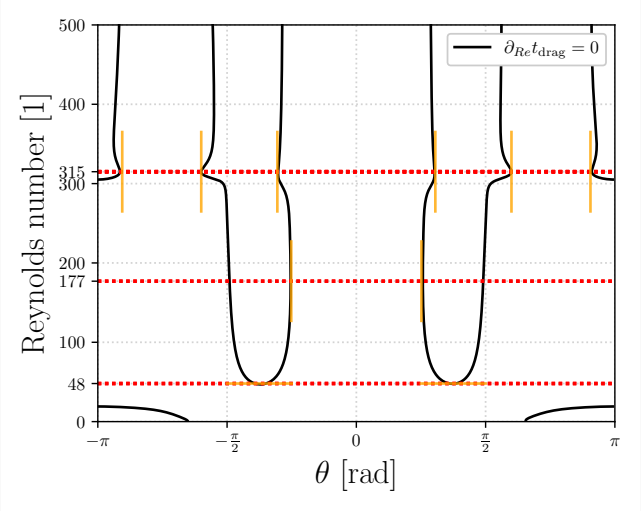
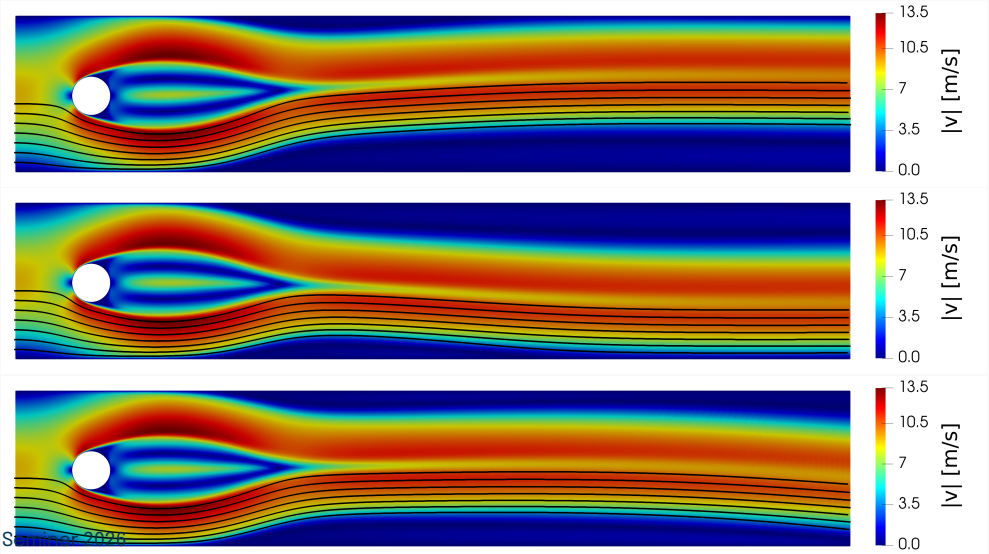


Figure: Pointwise drag: $\partial_{\mathrm{Re}} t_{\mathrm{drag}}(\theta, \mathrm{Re}) = 0$ shown in black

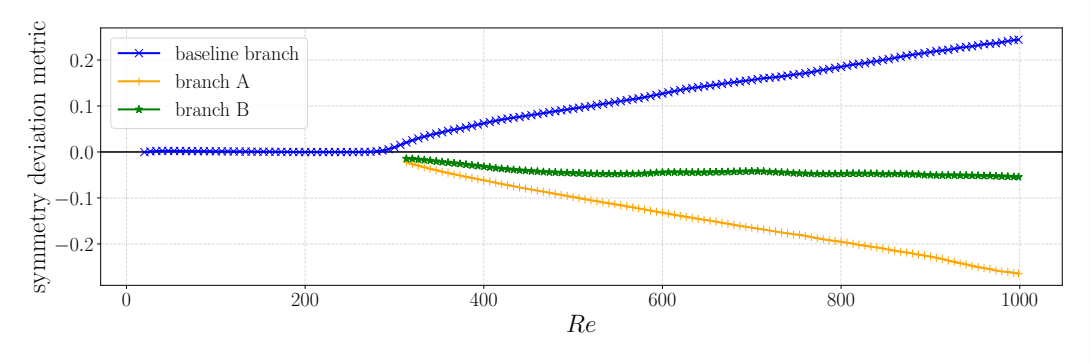
Multiple Steady Solutions at $\mathrm{Re} \geq 315$

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

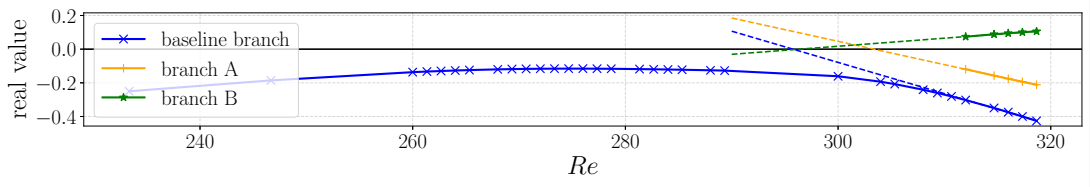


Bifurcation diagram

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University



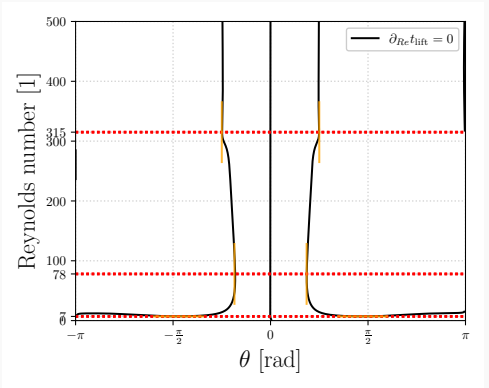
Maximal real eigenvalue behavior



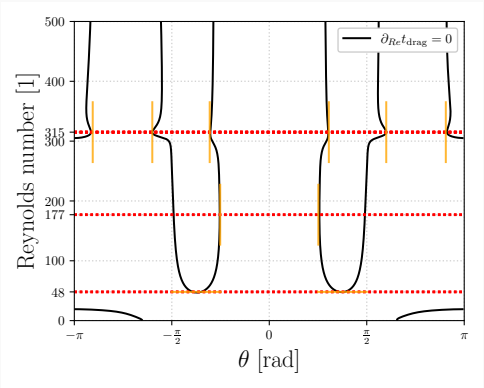
Multiple solutions animation

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Turning point – global view



(a) Pointwise lift: $\partial_{\mathrm{Re}} t_{\mathrm{lift}}(\theta, \mathrm{Re}) = 0$ in black



(b) Pointwise drag: $\partial_{\mathrm{Re}} t_{\mathrm{drag}}(\theta, \mathrm{Re}) = 0$ in black

Asymmetric shedding

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

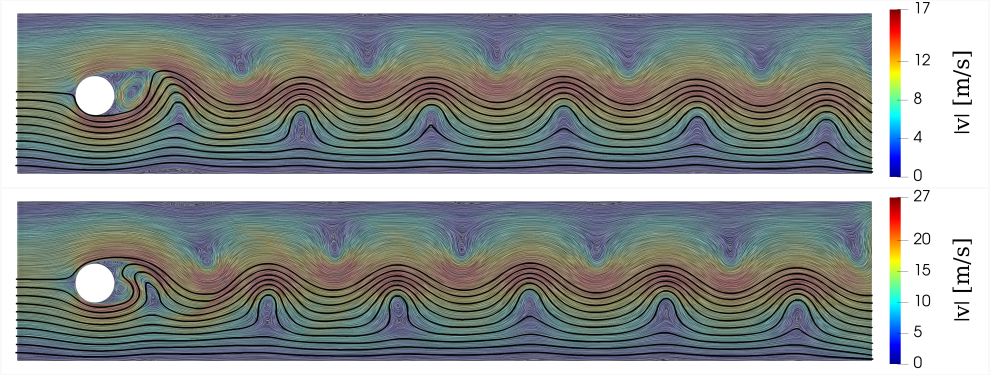


Figure: Non-symmetric decaying vortex street (depicted at $\mathrm{Re} = 670$) and von Karmán vortex street onset at $\mathrm{Re} = 950$. Latter one could not been reliably associated with a turning point.

Summary: All solutions sketch

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

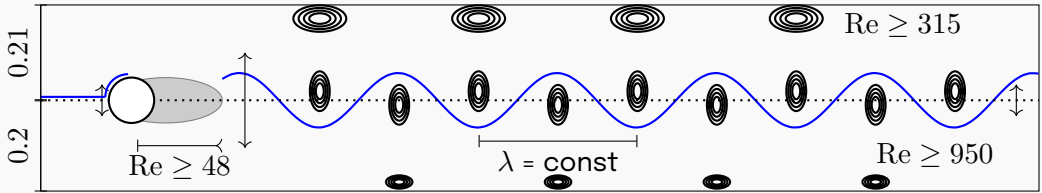
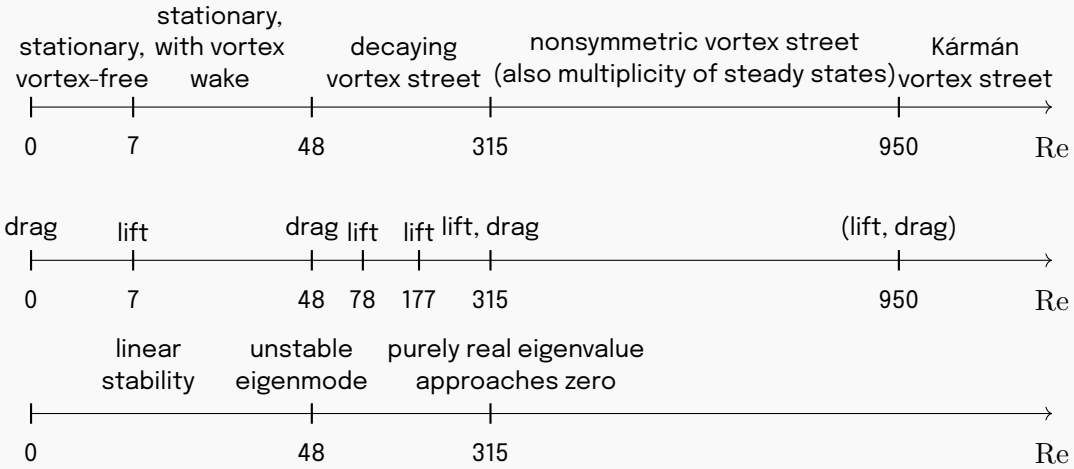


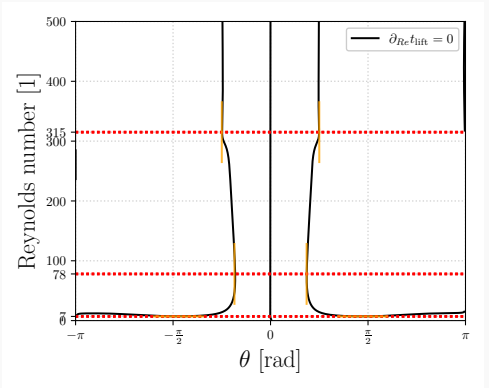
Figure: Schäfer-Turek benchmark. Sketch of the evolution of the unique time-periodic global-in-time attractor with increasing Reynolds number up to 1000. The arrows indicate trends with increasing Reynolds number. The onset Reynolds numbers of the key flow regimes correspond closely to characteristic changes in the pointwise traction profiles: (i) decaying vortex street at $\mathrm{Re} = 48$, (ii) saturation of wake width and onset of wall vortices at $\mathrm{Re} = 78$, (iii) non-symmetric time-periodic flow with larger vortices advected along the top wall (and still with decaying vortices along the centerline) at $\mathrm{Re} = 315$, and (iv) the emergence of the Kármán vortex street at $\mathrm{Re} = 950$.

Summary: Axis comparison

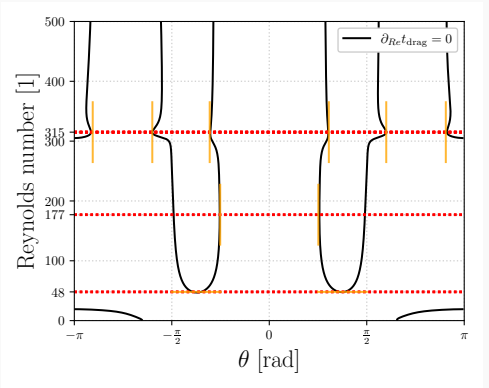


Turning point – global view

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University



(a) Pointwise lift: $\partial_{\mathrm{Re}} t_{\mathsf{lift}}(\theta, \mathrm{Re}) = 0$ in black



(b) Pointwise drag: $\partial_{\mathrm{Re}} t_{\mathsf{drag}}(\theta, \mathrm{Re}) = 0$ in black

Turning points empirical conditions

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

The neutral traction curves exhibit folds in Reynolds number, reminiscent of degeneracies in bifurcation theory but of derived quantity

$$t_c(\theta, \mathrm{Re}; (\mathbf{v}, p)(\mathrm{Re})),$$

rather than the solution map $(\mathbf{v}, p)(\mathrm{Re})$.

Neutral curve condition: $\partial_{\mathrm{Re}} \mathbf{t}(\theta, \mathrm{Re}) = 0$

Turning point types:

- **(A) Simple critical point**

$$\partial_{\mathrm{Re}} t_c = 0, \qquad \partial_\theta \partial_{\mathrm{Re}} t_c = 0$$

- **(B) Higher-order degeneracy**

$$\partial_{\mathrm{Re}} t_c = 0, \qquad \partial^2_{\mathrm{Re}} t_c = 0$$

Traction turning point framework

FACULTY OF MATHEMATICS AND PHYSICS
Charles University

Table: Observed correspondence between critical Reynolds numbers, turning-point types, and traction components.

Re	Flow transition / regime change	Type	Component	Symmetry content
7	Onset of steady vortices	A	Lift	Asymmetric
48	First Hopf bifurcation	A	Drag	Symmetric
78	Wake reorganization (2D wall effect / 3D mode E onset)	B	Lift	Asymmetric
177	Mode A instability onset	B	Drag	Symmetric
315	Attractor reorganization	B	Lift + multiple drag locations	Mixed

Turning point condition interpretation

FACULTY OF MATHEMATICS AND PHYSICS
Charles University

- **Type A turning points**
  - associated with the first appearance of a new flow structure
  - correspond to a simple critical point of the observable sensitivity
  - indicate a localized and non-degenerate response
- **Type B turning points**
  - associated with reorganization of already established unsteady dynamics
  - correspond to a higher-order degeneracy in Reynolds-number dependence
  - indicate a loss of regularity of the observable response
- **Lift component**
  - primarily reflects asymmetrical flow structures
  - is sensitive to symmetry-breaking effects
- **Drag component**
  - primarily reflects symmetric flow modifications
  - captures global streamwise force redistribution
- **Mixed drag/lift response**
  - indicates coexistence or interaction of multiple flow structures
  - suggests increasing spatial and dynamical complexity

Summary

FACULTY OF MATHEMATICS AND PHYSICS
Charles University

We observed:

- upstream footprint of flow dynamics on the obstacle boundary
- pointwise drag and lift traction as localized indicators of critical flow transitions
- correspondence between critical Reynolds numbers and turning points of traction sensitivity
- symmetry separation in the traction response: drag associated with symmetric flow modifications, lift with asymmetric structures
- (existence of a unique time-periodic global-in-time attractor)

References and acknowledgement

FACULTY OF MATHEMATICS AND PHYSICS Charles University

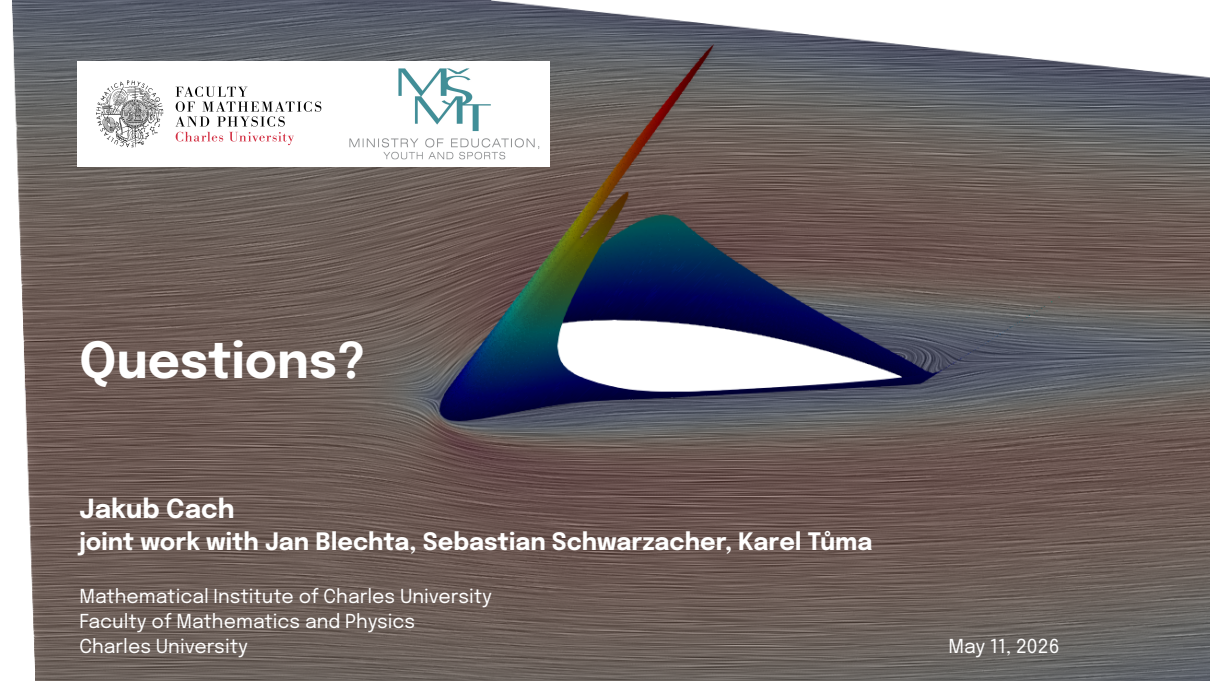
1) Turek, Schaefer; *Benchmark computations of laminar flow around cylinder, in Flow Simulation with High-Performance Computers II*, Notes on Numerical Fluid Mechanics 52, 547-566, Vieweg 1996

2) David A. Ham et al. *Firedrake User Manual*. First edition. Imperial College London and University of Oxford and Baylor University and University of Washington. May 2023. DOI: 10.25561/104839.

3) T. Horger, J. M. Melenk, and B. Wohlmuth, "On optimal L^2- and surface flux convergence in FEM," *Comput. Vis. Sci.*, vol. 16, no. 5, pp. 231–246, 2013. doi:10.1007/s00791-015-0237-z.

4) T. Apel, M. Mateos, J. Pfefferer, and A. Rösch, "Error estimates for Dirichlet control problems in polygonal domains: quasi-uniform meshes," *Math. Control Relat. Fields*, vol. 8, no. 1, pp. 217–245, 2018. doi:10.3934/mcrf.2018010.

- J.C., S.S. and K.T. have been supported by the ERC-CZ Grant LL2105 CONTACT. J.B., J.C. and K.T. thank for the support of the Charles University Research program No. UNCE/24/SCI/005. J.C. also thanks the Charles University Grant Agency for support, Grant No. 131124. S.S. and K.T. are Nečas Center for Mathematical Modeling members.



FACULTY OF MATHEMATICS AND PHYSICS
Charles University

MINISTRY OF EDUCATION, YOUTH AND SPORTS

Questions?

Jakub Cach
joint work with Jan Blechta, Sebastian Schwarzacher, Karel Tůma

Mathematical Institute of Charles University
Faculty of Mathematics and Physics
Charles University

May 11, 2026

Notion of a discrete pointwise traction

FACULTY OF MATHEMATICS AND PHYSICS
Charles University

Direct evaluation

$p_h,\ \mathbf{v}_h \mapsto \mathbf{t}_h := \mathbb{T}(p_h, \mathbf{v}_h)\mathbf{n}_h; \mathbb{T}(p, v) = -p\mathbb{I} + 2\mu\mathbb{D}$

Direct projection

$p_h,\ \mathbf{v}_h \mapsto \int_\Gamma \mathbf{t}_h \cdot \varphi \,\mathrm{d}\mathcal{S} := \int_\Gamma \mathbb{T}(p_h, \mathbf{v}_h)\mathbf{n}_h \cdot \varphi \,\mathrm{d}\mathcal{S}$

Variational approach

$p_h,\ \mathbf{v}_h \mapsto \int_\Gamma \mathbf{t}_\mathbf{n}^h \cdot \varphi_h \,\mathrm{d}\mathcal{S} := \int_\Omega \rho \frac{d\mathbf{v}_h}{dt} \cdot \varphi_h + \int_\Omega \mathbb{T}_h \cdot \nabla\varphi_h - \int_{\partial\Omega\setminus\Gamma} (\mathbb{T}_h\mathbf{n}_h) \cdot \varphi_h \,\mathrm{d}\mathcal{S} \qquad \forall \varphi_h \in V^h$

Discrete variational normal derivative ($\Delta u = f, u|_{\partial\Omega} = 0$)

$u_h \mapsto \int_{\partial\Omega} \partial_\mathbf{n}^h u_h\, v_h \,\mathrm{d}\mathcal{S} := \int_\Omega \nabla u_h \cdot \nabla v_h \,\mathrm{d}x - \langle f, v_h \rangle \qquad \forall v_h \in V^h$, solve for $\partial_\mathbf{n}^h u \in \mathrm{trace} V^h$

Pointwise traction theorems

FACULTY OF MATHEMATICS AND PHYSICS
Charles University

Theorem (Horger, Melenk, and Wohlmuth [3, Corollary 5.3])

Suppose that Poisson eq. admits $H^{3/2+\varepsilon}$ regularity, i.e., there exist $\varepsilon > 0$ and $c > 0$ such that the solution operator $T\colon f \mapsto u$ for Poisson eq. satisfies

$$\|Tf\|_{H^{3/2+\varepsilon}(\Omega)} \leq c\|f\|_{(H_0^{1/2-\varepsilon}(\Omega))'}.$$

Then there exists $C > 0$ such that

$$\|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u_h\|_{L^2(\partial\Omega)} \leq C(1 + \delta_{k,1}|\log h|) \begin{cases} h^k \|u\|_{B^{k+3/2}_{2,1}(\Omega)}, \\ h^s \|u\|_{B^{s+3/2}_{2,\infty}(\Omega)}, \quad s \in (0,k), \\ \|u\|_{B^{3/2}_{2,1}(\Omega)}. \end{cases}$$

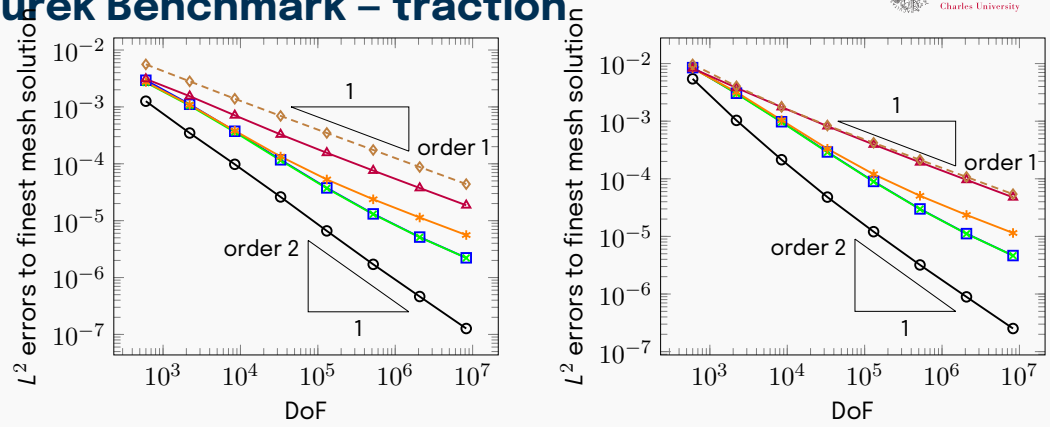
Advantages

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

- variational defined already for H^1 (otherwise $u \in H^{3/2} \implies \partial_{\mathbf{n}} u \in L^2(\partial\Omega)$)
- circumvents error on curved boundary (no $\mathbf{n}$)
- robust – best convergence in all geometric and regularity conditions
- possibility of super-convergence (+1/2)
- cheap postprocessing (sparsity of φ)
- message for FSI: do coupling weakly

Turek Benchmark – traction

FACULTY OF MATHEMATICS AND PHYSICS
Charles University



(a) Stokes

(b) Navier-Stokes (original benchmark)

Variational traction: CG(1)
Projected: CG(1), analytical $\mathbf{n}$
Projected: CG(1), CG(1) $\mathbf{n}_h$
Direct: DG(1), analytical $\mathbf{n}$
Direct: DG(1), CG(1) $\mathbf{n}_h$
Direct: DG(1), mesh facet $\mathbf{n}_h$