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A thermodynamically consistent Johnson–Segalman–Giesekus model

numerical simulation of the rod climbing effect

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joint work with Patrick Farrell, Josef Málek, Karel Tůma

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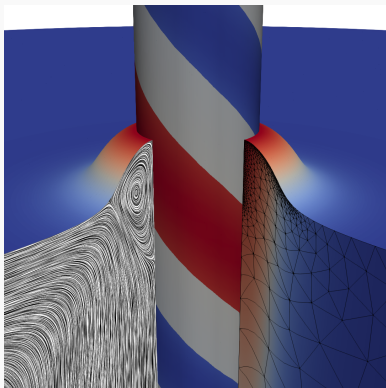
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J. Cach, P. E. Farrell, J. Málek, K. Tůma: A thermodynamically consistent Johnson–Segalman–Giesekus model: numerical simulation of the rod climbing effect, Appl. Eng. Sci. 26 (2026), 100315. doi:10.1016/j.apples.2026.100315

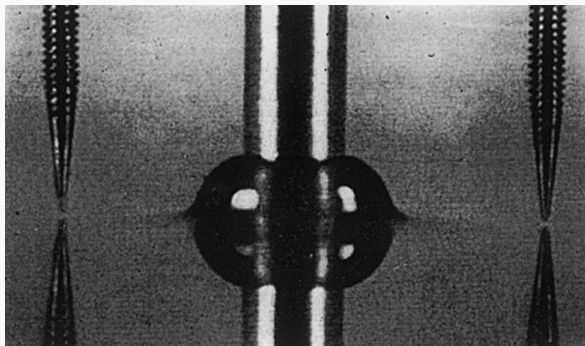
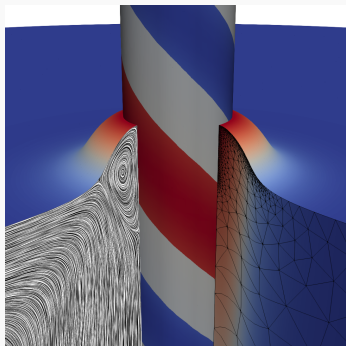
Thermodynamically consistent Johnson–Segalman–Giesekus model



- Rod climbing phenomenon
- Rise of a fluid along a rod rotating about its axis
- Presence of normal stress differences in the model

$$\begin{aligned} \mathbb{T} &= -p\mathbb{I} + 2\mu_s \mathbb{D} + \alpha G(\mathbb{B} - \mathbb{I}) \\ \nabla_a \mathbb{B} + \frac{1}{\tau} \left((1 - \alpha)(\mathbb{B} - \mathbb{I}) + \alpha(\mathbb{B}^2 - \mathbb{B}) \right) &= \mathbb{O} \end{aligned}$$

Rod climbing experiments



G. S. Beavers, D. D. Joseph: The rotating rod viscometer, *Journal of Fluid Mechanics* 69(3), 475–511 (1975). doi:10.1017/S002211207500153X.

Johanson–Segalman–Giesekus models

- Thermodynamically consistent JSG: Model I

$$\begin{aligned}\mathbb{T} &= -p\mathbb{I} + 2\mu_s\mathbb{D} + \alpha G(\mathbb{B} - \mathbb{I}), \\ \nabla_{\sigma} \mathbb{B} + \frac{1}{\tau} \left((1 - \alpha)(\mathbb{B} - \mathbb{I}) + \alpha(\mathbb{B}^2 - \mathbb{B}) \right) &= \mathbb{O}\end{aligned}$$



M. Dostálík, V. Průša, T. Skřivan: On diffusive variants of some classical viscoelastic rate-type models, AIP Conference Proceedings **2107**(1). (2019). doi:10.1063/1.5099919.

- Popular engineering JSG: Model II

$$\begin{aligned}\mathbb{T} &= -p\mathbb{I} + 2\mu_s\mathbb{D} + G(\mathbb{B} - \mathbb{I}), \\ \nabla_{\sigma} \mathbb{B} + \frac{1}{\tau} \left((1 - \alpha)(\mathbb{B} - \mathbb{I}) + \alpha(\mathbb{B}^2 - \mathbb{B}) \right) &= 2(1 - \alpha)\mathbb{D}\end{aligned}$$



M. W. Johnson, D. Segalman: A model for viscoelastic fluid behavior which allows non-affine deformation, Journal of Non-Newtonian Fluid Mechanics (1977).

Gordon–Schowalter objective derivatives

- Defined as

$$\overset{\nabla_a}{\dot{\mathbf{A}}} = \dot{\mathbf{A}} - a(\mathbf{D}\mathbf{A} + \mathbf{A}\mathbf{D}) - (\mathbf{W}\mathbf{A} - \mathbf{A}\mathbf{W}), \quad a \in [-1, 1]$$

- Kinematics allow it
- Heuristic microscopic reasoning: slip between polymer chains and velocity field
- In practice: allows to fit experiments
- Rajagopal & Srinivasa framework leads to the upper convected derivative

TD consistency results

- Reduced thermodynamic identity (isothermal)

$$\xi = \mathbb{T} : \mathbb{D} - \rho \dot{\psi} \quad \text{with} \quad \xi \geq 0$$

- Model I:

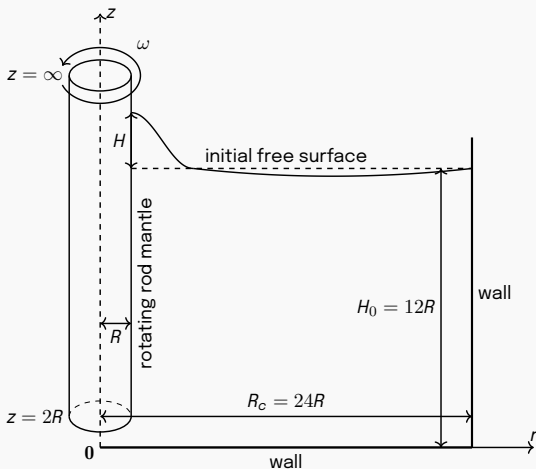
$$\xi = 2\mu_s |\mathbb{D}|^2 + (1 - \alpha) \frac{G}{2\tau} (\mathbb{I} - \mathbb{B}^{-1}) : (\mathbb{B} - \mathbb{I}) + \alpha \frac{G}{2\tau} |\mathbb{B} - \mathbb{I}|^2 \geq 0$$

- Model II:

$$\xi = 2\mu_s |\mathbb{D}|^2 + (1 - \alpha) \frac{G}{2\tau} (\mathbb{I} - \mathbb{B}^{-1}) : (\mathbb{B} - \mathbb{I}) + \alpha \frac{G}{2\tau} |\mathbb{B} - \mathbb{I}|^2 + \textcolor{red}{G(1 - \alpha)(\mathbb{B} + \mathbb{B}^{-1}) : \mathbb{D}}$$

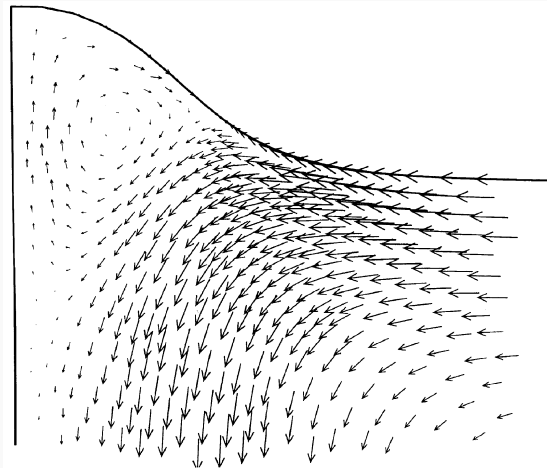
Numerical simulations ingredients

- Rod-climbing geometry (fixed)
- Fluid properties (fixed)
- Objective param. α varied
- JS or Giesekus model
- ALE method (Lagrange free-surface condition)
- Firedrake (FEM library)
- Results



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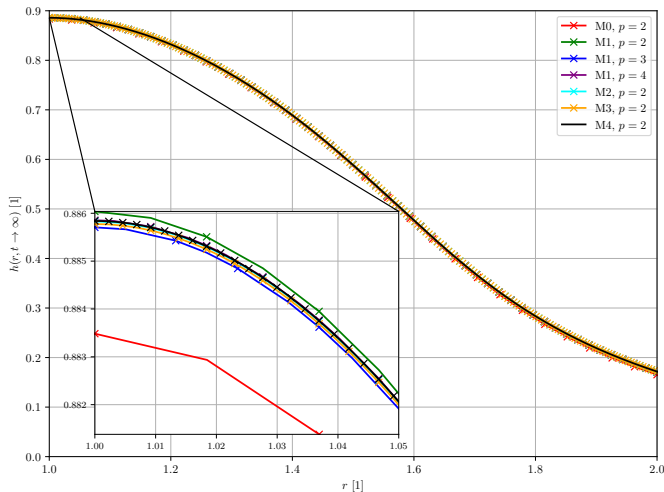
System to solve

$$\begin{aligned}
 & \int_{\Omega_\chi} r \hat{J} \operatorname{tr} \left((\nabla_{\hat{\chi}} \mathbf{v}) \hat{\mathbb{F}}^{-1} \right) \psi = 0, \\
 & \operatorname{Re} \int_{\Omega_\chi} r \hat{J} \left[\frac{\partial \mathbf{v}}{\partial t} + (\nabla_{\hat{\chi}} \mathbf{v}) \left(\hat{\mathbb{F}}^{-1} \left(\mathbf{v} - \frac{\partial \hat{\mathbf{u}}}{\partial t} \right) \right) \right] \cdot \phi + \int_{\Omega_\chi} r \hat{J} \hat{\mathbb{T}} \hat{\mathbb{F}}^{-T} \cdot \nabla_{\hat{\chi}} \phi \\
 & + \frac{1}{Ca} \int_{\Gamma_\chi^{\text{free}}} r \|\hat{\mathbb{F}}^{-T} \mathbf{n}\| \left[\mathbb{I} - \frac{\hat{\mathbb{F}}^{-T} \mathbf{n}}{|\hat{\mathbb{F}}^{-T} \mathbf{n}|} \otimes \frac{\hat{\mathbb{F}}^{-T} \mathbf{n}}{|\hat{\mathbb{F}}^{-T} \mathbf{n}|} \right] (\nabla_{\hat{\chi}} \phi) \hat{\mathbb{F}}^{-1} + \frac{1}{St} \int_{\Omega_\chi} r \hat{J} \mathbf{e}_z \cdot \phi = 0, \\
 & \hat{\mathbb{T}} = -p \mathbb{I} + 2 \frac{\mu_s}{\mu_s + \mu_p} \mathbb{D}_{\hat{\chi}} + \frac{\alpha}{We} \frac{\mu_p}{\mu_s + \mu_p} (\mathbb{B} - \mathbb{I}), \\
 & \int_{\Omega_\chi} r \hat{J} \left. \nabla_a \mathbb{B} \right|_{\hat{\chi}} \cdot \mathbb{A} + r \hat{J} \left[\frac{1-\alpha}{We} (\mathbb{B} - \mathbb{I}) + \frac{\alpha}{We} (\mathbb{B}^2 - \mathbb{B}) \right] \cdot \mathbb{A} = 0, \\
 & \int_{\Omega_\chi} r \partial_z \hat{u}_z \partial_z \hat{w}_z - \int_{\partial \Omega_\chi} r (\partial_z \hat{u}_z) \hat{w}_z - \int_{\partial \Omega_\chi} r \left[\frac{\partial \hat{u}_z}{\partial t} + (v_r \partial_r \hat{u}_z - v_z) \right] \left(\partial_z \hat{w}_z - \frac{\beta_{st}}{h_{\min}} \hat{w}_z \right) = 0,
 \end{aligned}$$

Implementation

- Finite element method using the Firedrake library
- High-order discretization ($p = 4$): Scott-Vogelius pair $([CG_p]^3 - DG_{p-1})$
- Matching high-order spaces: $[CG_{p-1}]_{\text{sym}}^{3 \times 3}$ for extra stress, CG_p for mesh displacement
- High-order geometry of free surface preserved via virtual-ALE formulation
- Second-order three-step θ -scheme (Glowinski) with excellent volume preservation
- Fully monolithic coupling (no splitting, no additional stabilization required)
- Efficient nonlinear solution: lagged-Jacobian strategy + LU reuse
- High accuracy at moderate cost: $\sim 130\text{k}$ DoF, parallel, minutes on a laptop
- No interface reconstruction or CFL restrictions

Mesh-convergence test (Oldroyd-B)

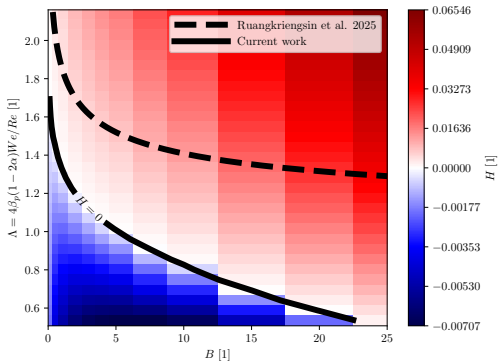


Comparison with analytical solution ($a = 1$)

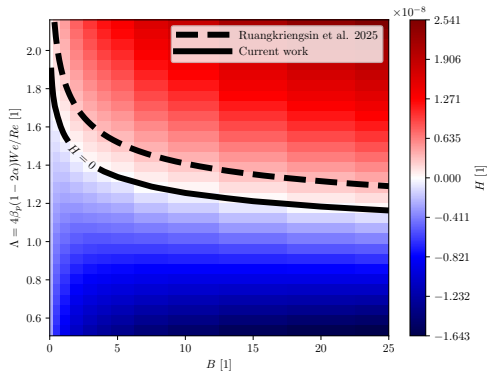


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- Onset of rod-climbing effect for Giesekus model

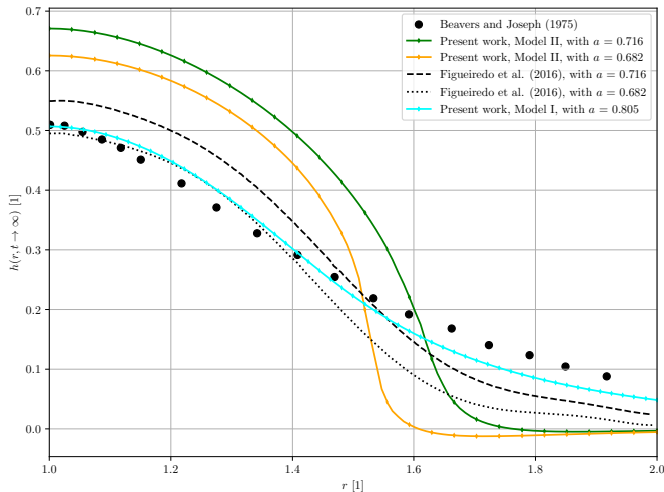


(a) $We = 0.195$, $\mathcal{G} = 0.069$



(b) $We = 9.73 \times 10^{-3} \ll 1$, $\mathcal{G} = 138 \gg 1$

Comparison with experiments and simulations



Summary

- ✓ Model I satisfies the 2nd law of thermodynamics
- ✓ Model I shows superior agreement with experiments
- ✗ Model II violates the 2nd law of thermodynamics
- ✓ Efficient and accurate code