On Bifurcations and Traction Forces on an Obstacle in Incompressible Flow

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Introduction

We present a systematic numerical investigation of bifurcations in the two-dimensional incompressible Navier–Stokes flow past a confined circular cylinder. In particular, we analyse the relation between traction profiles and qualitative changes in its long-time flow behavior. The results indicate a clear correspondence between changes in the traction profiles of the steady Navier–Stokes equations and bifurcations of the unsteady Navier–Stokes equations, including the manifestation of symmetry breaking, oscillations, and multiple steady solutions. While long-time flow behavior is often explored through direct numerical simulation of the unsteady equations, which is accurate but computationally demanding, these results suggest a more inexpensive strategy to detect critical Reynolds numbers (Re).

Problem description

Unsteady incompressible Navier-Stokes equations with inflow and outflow:

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega,\\
\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) &= \operatorname{div} \mathbb{T} && \text{in } \Omega,\\
\mathbb{T} &= -p\mathbb{I} + \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^T) && \text{in } \Omega,\\
\mathbf{v} &= \mathbf{v}_D^i && \text{on } \Gamma_i \subset \partial\Omega,\\
\mathbb{T}\mathbf{n} &= \mathbf{0} && \text{on } \partial\Omega \setminus \cup_i \Gamma_i.
\end{aligned}
$$

Theoretical background

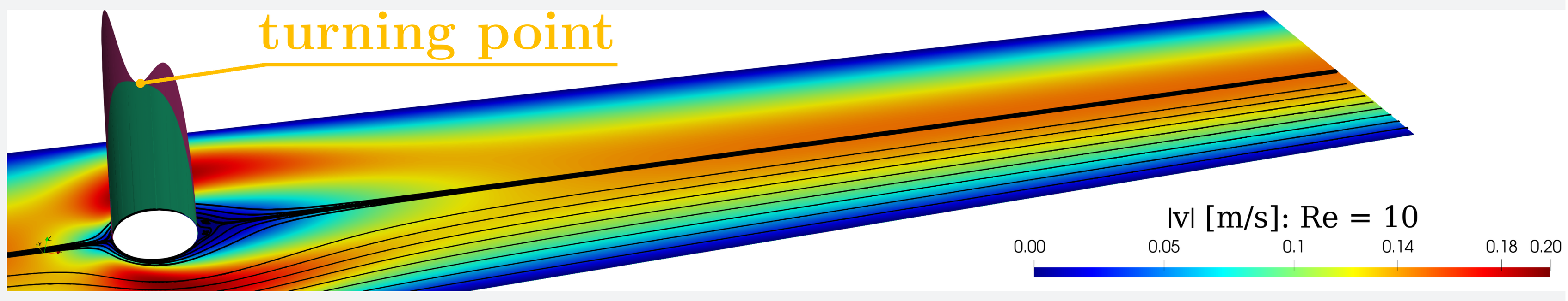
- Steady BVP not unique in 2D, 3D (except low Re)
- IBVP unique in 2D (Ladyženskaja)
- For single-obstacle flows at low Re, the steady solution is the unique attractor (no time-periodic states exist) in both 2D and 3D
- Key questions:
  - What ensures stability of a solution?
  - Are long-time attractors unique?
  - Are eigenvalues decisive?
  - Do bifurcations leave a footprint in the steady traction?

Numerical methods & Imeplementation

- Firedrake [1] finite element framework with PETSc backend
- Mesh generation: Netgen (curved boundaries), Gmsh
- Spatial discretization: Taylor–Hood elements (P2–P1)
- Time integration: Crank–Nicolson scheme
- Pointwise traction: variationally computed on boundary layer mesh
- Deflated continuation for multiple steady branches: DefCon
- Eigenvalues: Linear Perturbation operator: GNHEP solver in SLEPc
- Geometrical setup: Turek–Schäfer Benchmark flow-around-cylinder [2]

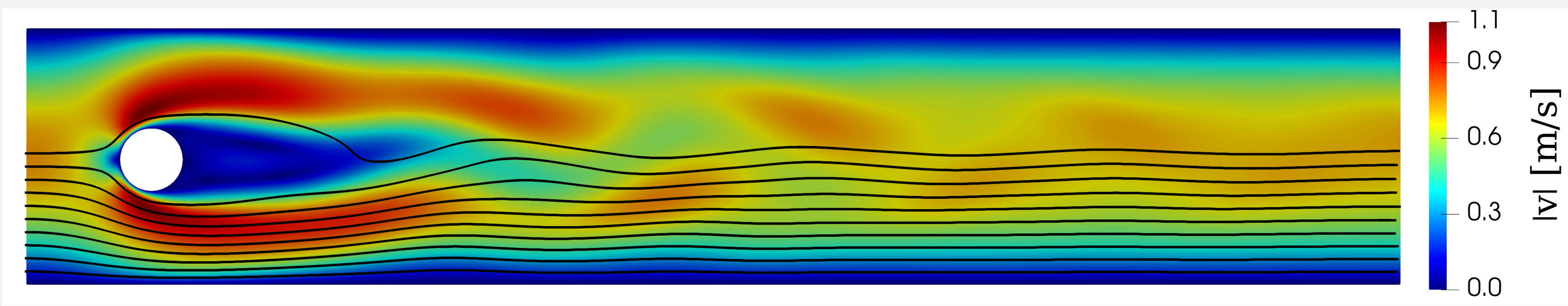
Onset of steady vortices ($\mathrm{Re} \doteq 7$)

The laminar flow loses stability, accompanied by a turning point in the pointwise lift as a function of Re.



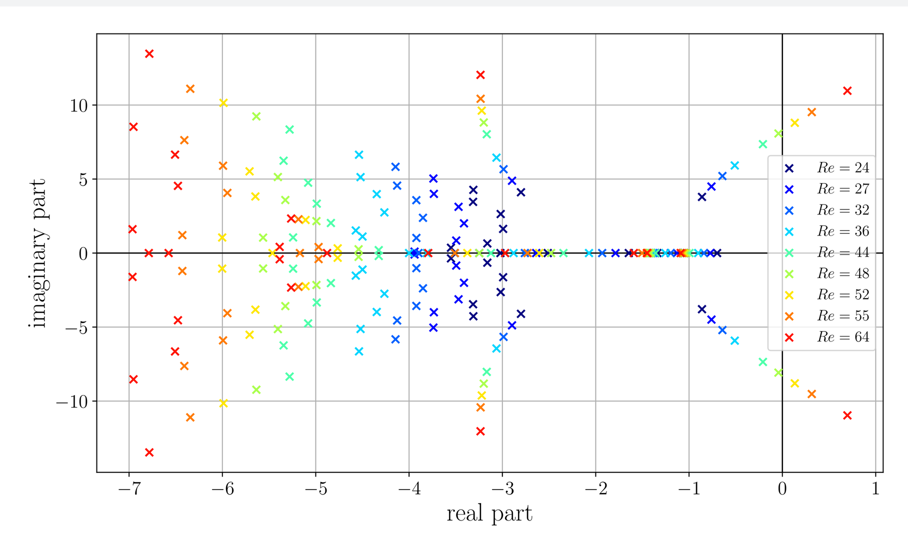
Onset of periodic laminar shedding ($\mathrm{Re} \doteq 48$)

Time-stepping from the steady solution, the flow develops small-amplitude, time-periodic oscillations.



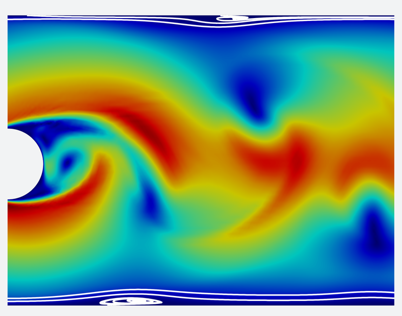
Linear (in)stability ($\mathrm{Re} \doteq 48$)

Eigenvalues of the Linear Perturbation Operator indicate the onset of a Hopf bifurcation in the unsteady Navier–Stokes equations.



Wake width saturation ($\mathrm{Re} \doteq 78$)

The oscillatory wake widens with increasing Re until vortices form near the walls and the width saturates. These wall vortices are then advected downstream by the main flow.

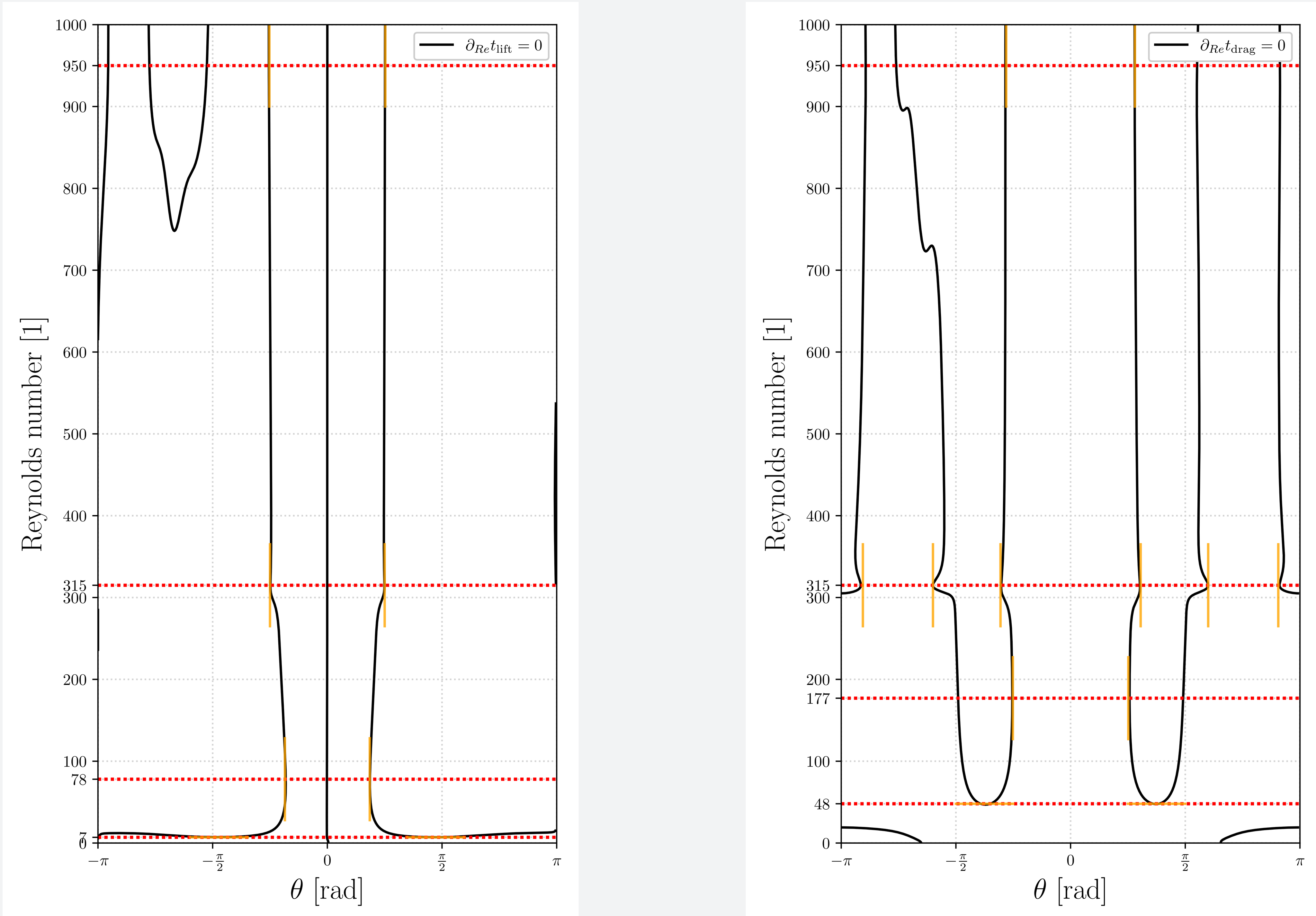


3D instability onset ($\mathrm{Re} \doteq 177$)

In 3D, secondary instabilities appear for $Re \doteq 170$–200. The classical Mode A instability can be viewed as a spanwise modulation of the 2D wake: each streamwise slice resembles the 2D pattern, but with a phase shift that varies along the spanwise direction, i.e., perpendicular to our 2D setup.

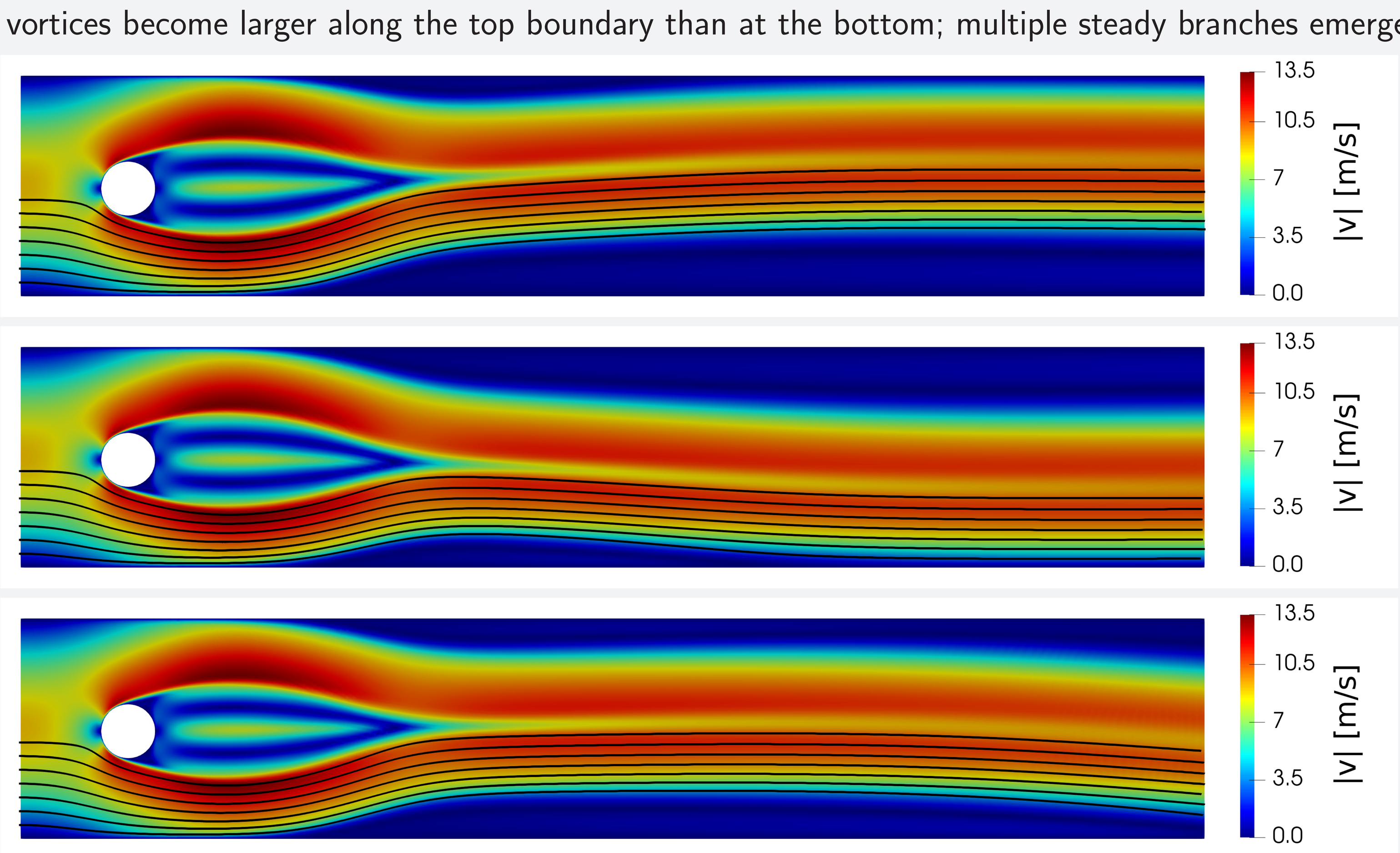
Pointwise lift and drag turning points $\mathrm{Re} \in (0, 1000)$

Implicit curves defined by $\partial_{\mathrm{Re}} t_{\text{drag,lift}}(\theta, \mathrm{Re}) = 0$ along the cylinder boundary $\theta \in (-\pi, \pi)$, computed from the traction profiles of steady solutions. The folds of the black curve, highlighted by short orange tangent lines, correspond to **turning points**, whose Re values are marked by red lines.



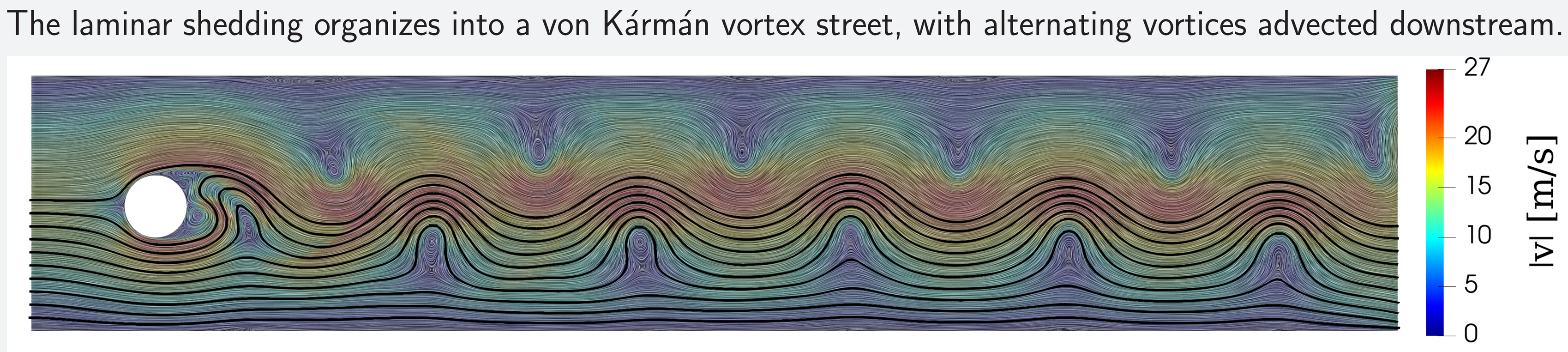
(Un)steady symmetry breaking ($\mathrm{Re} \doteq 315$)

Wall vortices become larger along the top boundary than at the bottom; multiple steady branches emerge.



Onset of periodic vortex shedding ($\mathrm{Re} \doteq 950$)

The laminar shedding organizes into a von Kármán vortex street, with alternating vortices advected downstream.



References & Acknowledgement

[1] David A. Ham et al. *Firedrake User Manual*. First edition. Imperial College London and University of Oxford and Baylor University and University of Washington. May 2023. DOI: 10.25561/104839.

[2] Turek, Schäfer; *Benchmark computations of laminar flow around cylinder; in Flow Simulation with High-Performance Computers II*, Notes on Numerical Fluid Mechanics 52, 547-566, Vieweg 1996

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