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On bifurcations and traction forces on an obstacle in incompressible flow

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joint work with Jan Blechta, Sebastian Schwarzacher, Karel Tůma

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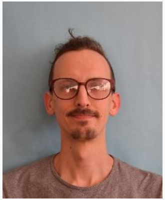


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Jakub Cach, Karel Tůma, Jan Blechta, & Sebastian Schwarzacher, *On bifurcations and traction forces on an obstacle in incompressible flow*, **arXiv:2512.15424 [physics.flu-dyn] (2025)**

Aim

- Traction: transfer of momentum between fluid–obstacle
- **Q:** *Can a steady boundary observable, namely the pointwise traction on the obstacle, indicate bifurcations of the unsteady Navier–Stokes dynamics?*
- Systematically observe all transition in the flow up to $\mathrm{Re} = 500$
- A posteriori connect bifurcations to traction changes
- Critical points in the traction $\partial_{\mathrm{Re}} \mathbf{t} = 0$

Equations and traction

- Unsteady incompressible Navier-Stokes equations

$$\rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = \text{div } \mathbb{T} \quad \text{in } \Omega,$$
$$\text{div } \mathbf{v} = 0 \quad \text{in } \Omega,$$
$$\mathbb{T} = -p\mathbb{I} + \mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T),$$
$$\mathbf{v} = \mathbf{v}_D^i \quad \text{on } \Gamma_i \subset \partial\Omega,$$
$$\mathbb{T}\mathbf{n} = 0 \quad \text{on } \partial\Omega \backslash \mathrm{U}\Gamma_i.$$

- Traction

$$\boxed{\mathbf{t}(\theta; \mathrm{Re}) := \mathbb{T}\mathbf{n}.}$$

- Evaluation of pointwise traction

$$p_h, \mathbf{v}_h \mapsto \int_\Gamma \mathbf{t}_\mathbf{n}^h \cdot \varphi_h \, \mathrm{d}S := \int_\Omega \rho \frac{d\mathbf{v}_h}{dt} \cdot \varphi_h + \int_\Omega \mathbb{T}_h \cdot \nabla \varphi_h - \int_{\partial\Omega\backslash\Gamma} (\mathbb{T}_h \mathbf{n}_h) \cdot \varphi_h \, \mathrm{d}S \qquad \forall \varphi_h \in V^h$$

Schäfer–Turek benchmark

- $t_{\mathsf{drag}}(\theta; \mathrm{Re}) := t_x(\theta; \mathrm{Re})$
- $t_{\mathsf{lift}}(\theta; \mathrm{Re}) := t_y(\theta; \mathrm{Re})$

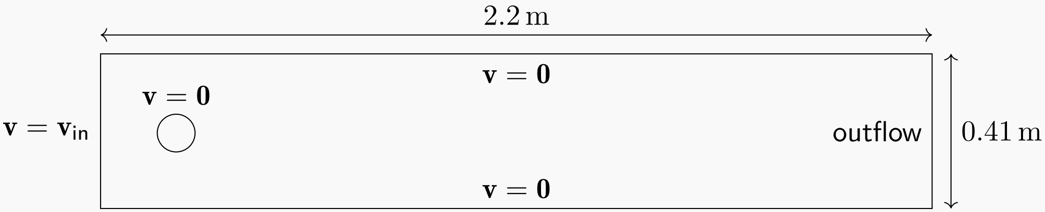


Figure: The Schäfer–Turek benchmark. The lower left corner is at $(0\,\mathrm{m}, 0\,\mathrm{m})$ and the cylinder center is located at $(0.2\,\mathrm{m}, 0.2\,\mathrm{m})$ with radius $R = 0.05\,\mathrm{m}$. The domain is slightly asymmetric. The benchmark does not require use of a specific outflow condition.

Navier-Stokes theory — What to expect

- steady BVP non-unique in 2D, 3D (except low Re)
- IBVP unique in 2D (Ladyženskaja)
- Uniqueness of time-periodic solutions in 2D, 3D
    - Steady solution only unique time-per. solution in case of a single obstacle flow (low Re)
- Open in rigorous manner:
    - What makes stable solution?
    - Are long-time attractors unique?
    - Are eigenvalues the key?

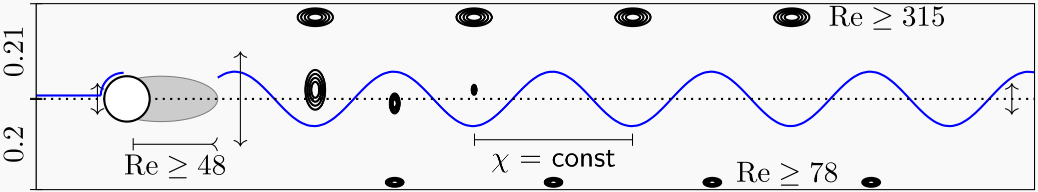
How we numerically analyze

- Discretization: Firedrake; Taylor-Hood CG(2,1); monolitic; Newton+LU; Crank-Nicolson
- global velocity field of the unsteady solution (long-time attractor)
- global velocity field of the steady solution
- traction profile of the steady solution (computed using dual method)
- eigenvalues of the Linear Perturbation Theory (LPT) operator (2D and 3D variant)
- deflated continuation of steady equations

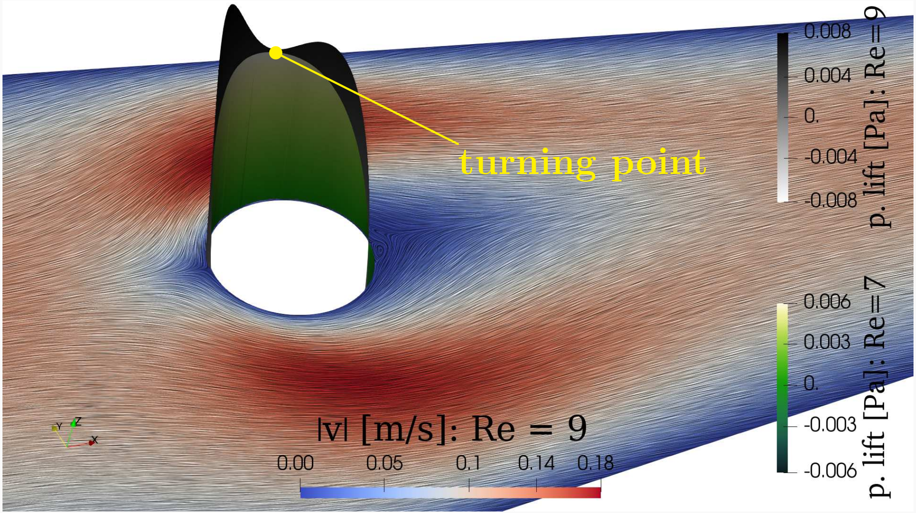
Long-time solutions for $\mathrm{Re} \in (0, 500)$

stationary, vortex-free	stationary, with vortex wake	decaying vortex street	nonsymmetric vortex street (also multiplicity of steady states)

0 — 7 — 48 — 315 — Re



Pointwise traction



Critical curves → Turning points: global view

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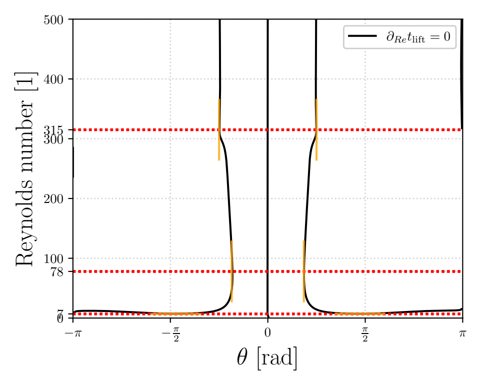
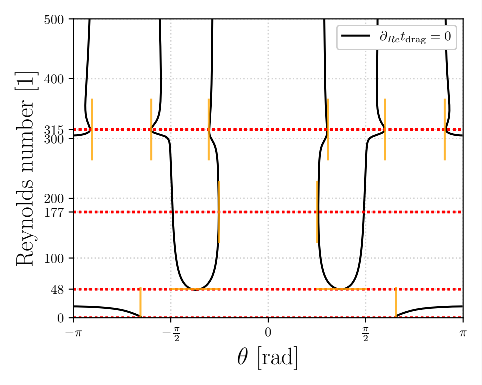


(a) Pointwise lift: $\partial_{\mathrm{Re}} t_{\text{lift}}(\theta, \mathrm{Re}) = 0$ in black



(b) Pointwise drag: $\partial_{\mathrm{Re}} t_{\text{drag}}(\theta, \mathrm{Re}) = 0$ in black

Turning points empirical conditions

The neutral traction curves exhibit folds in Reynolds number, reminiscent of degeneracies in bifurcation theory but of derived quantity

$$t_c(\theta, \mathrm{Re}; (\mathbf{v}, p)(\mathrm{Re})),$$

rather than the solution map $(\mathbf{v}, p)(\mathrm{Re})$.

Neutral curve condition: $\partial_{\mathrm{Re}} \mathbf{t}(\theta, \mathrm{Re}) = 0$

Turning point types:

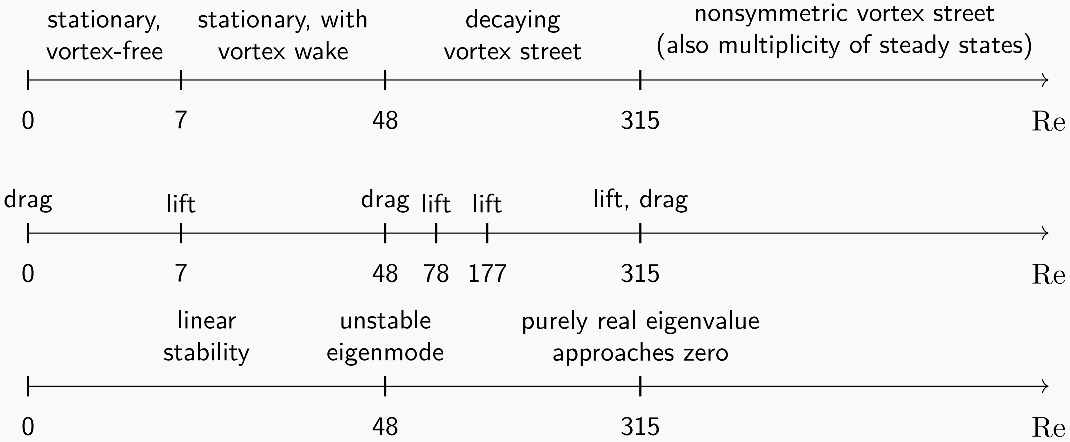
- **(A) Horizontal (simple critical point)**

$$\partial_{\mathrm{Re}} t_c = 0, \qquad \partial_\theta \partial_{\mathrm{Re}} t_c = 0$$

- **(B) Vertical (higher-order degeneracy)**

$$\partial_{\mathrm{Re}} t_c = 0, \qquad \partial^2_{\mathrm{Re}} t_c = 0$$

Summary: Axis comparison



Summary

We observed:

- An **upstream** footprint of flow dynamics on the obstacle boundary
- Pointwise drag and lift traction as localized **indicators** of flow transitions

References and acknowledgement

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1) Turek, Schaefer; *Benchmark computations of laminar flow around cylinder; in Flow Simulation with High-Performance Computers II*, Notes on Numerical Fluid Mechanics 52, 547-566, Vieweg 1996

2) David A. Ham et al. *Firedrake User Manual*. First edition. Imperial College London and University of Oxford and Baylor University and University of Washington. May 2023. DOI: 10.25561/104839.

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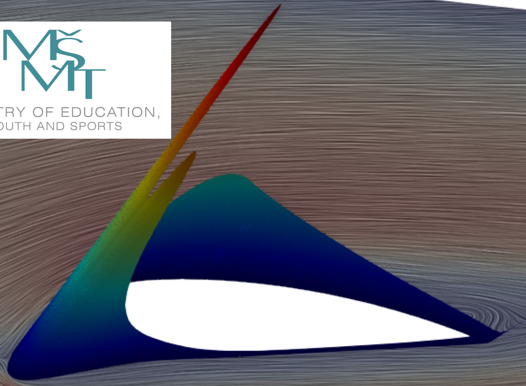

Questions?

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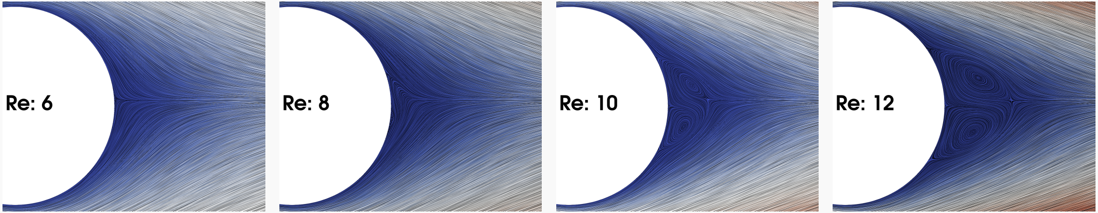
Results: All transitions found

- Low Re flow ($\mathrm{Re} \lesssim 7$)



(a) $\mathrm{Re} = 6$ (b) $\mathrm{Re} = 8$ (c) $\mathrm{Re} = 10$ (d) $\mathrm{Re} = 12$

Figure: Velocity field on the downstream side of the cylinder in line integral convolution plot (streamlines). A pair of stationary vortices appears at $\mathrm{Re} = 7$ and grows in size with increasing Re.

Results: All transitions found

- Non-trivial long-time solution $(\mathrm{Re} \geq 48)$

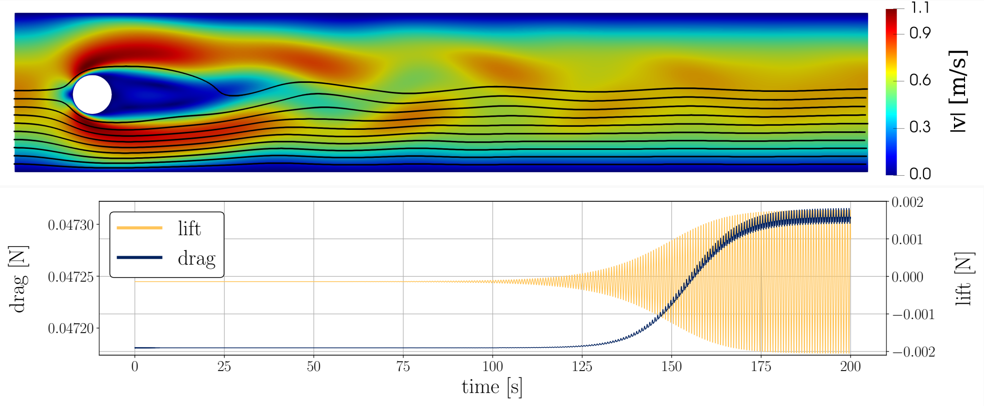
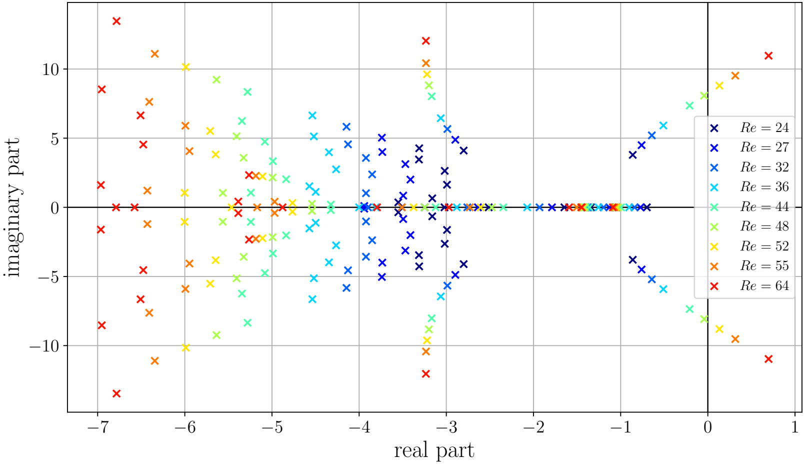


Figure: Numerical simulation of evolution problem at $\mathrm{Re} = 50$.

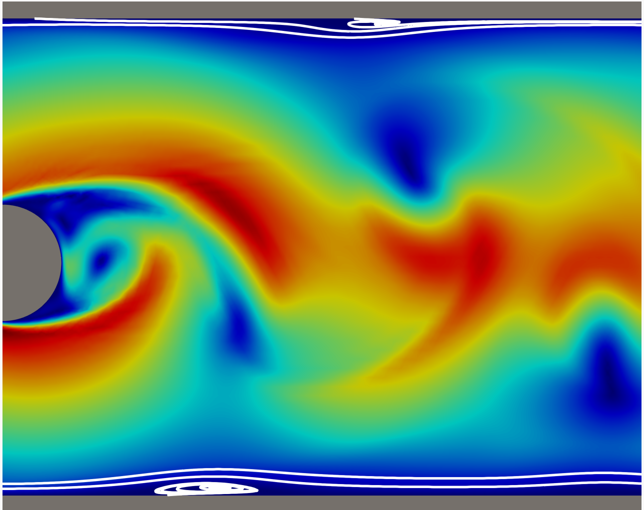
Results: All transitions found

- Hopf bifurcation ($\mathrm{Re} = 48$)



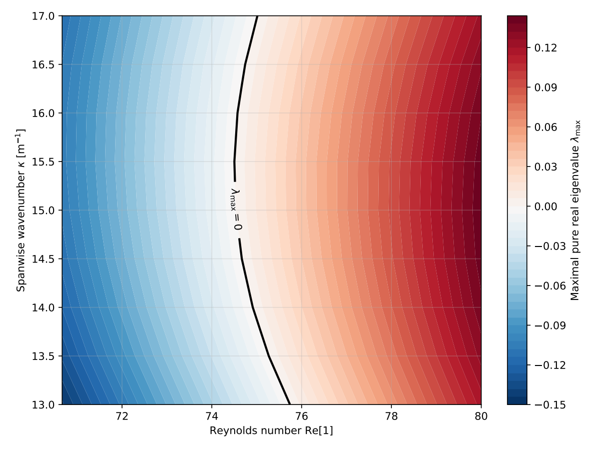
Results: All transitions found

- Wall effect: vortices along boundaries ($\mathrm{Re} \geq 78$)



Results: All transitions found

- Steady 3D bifurcation: Mode E $(\mathrm{Re} = 75)$

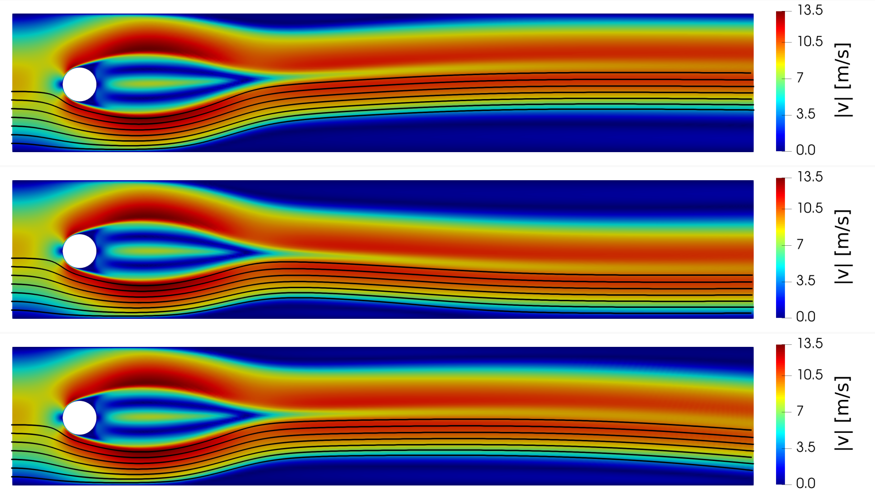


- 3D perturbation of 2D base flow
- spatial wavelength κ in spanwise coord. z must be solved for
- observation: spatial period $\approx$ domain width

$$
\begin{aligned}
v_x &= v_{\mathrm{st},x}(x,y) + \varepsilon e^{\lambda t}\,\hat{v}_x(x,y)\cos(\kappa z),\\
v_y &= v_{\mathrm{st},y}(x,y) + \varepsilon e^{\lambda t}\,\hat{v}_y(x,y)\cos(\kappa z),\\
v_z &= \varepsilon e^{\lambda t}\,\hat{v}_z(x,y)\sin(\kappa z),\\
p &= p_{\mathrm{st}}(x,y) + \varepsilon e^{\lambda t}\,\hat{p}(x,y)\cos(\kappa z)
\end{aligned}
$$

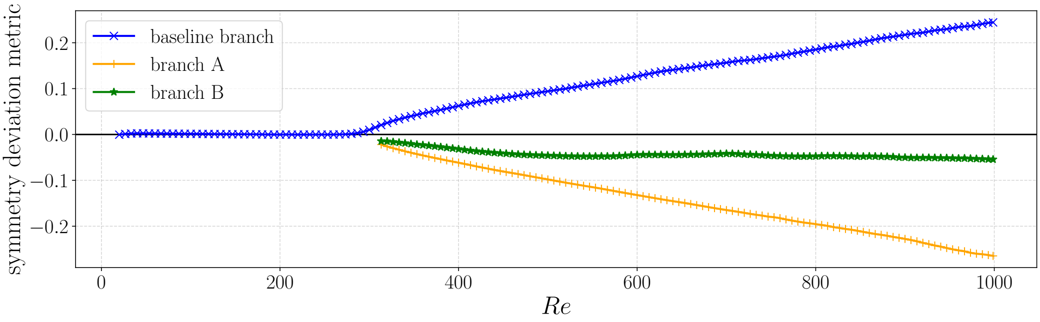
Results: All transitions found

- Multiple steady solutions at $\mathrm{Re} \geq 315$



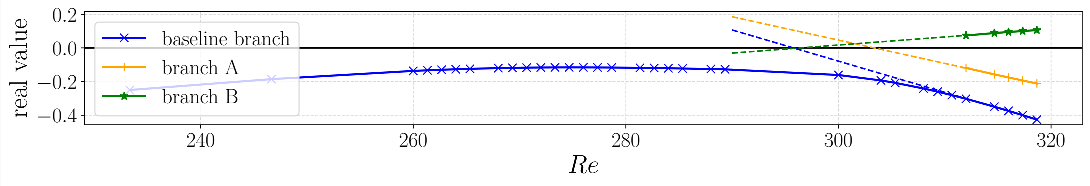
Results: All transitions found

- Bifurcation diagram for steady state multiplicity $(\mathrm{Re} = 315)$



Results: All transitions found

- Maximal real eigenvalue behavior



Results: All transitions found

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- Asymmetric shedding along top wall $\mathrm{Re} \geq 315$

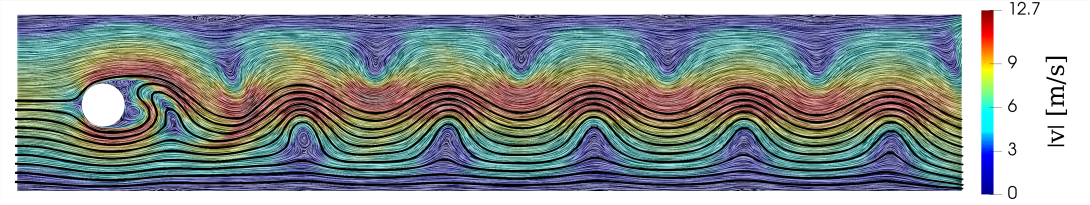

Figure: Non-symmetric decaying vortex street depicted at $\mathrm{Re} = 500$.

Notion of a discrete pointwise traction

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Direct evaluation

$p_h,\ \mathbf{v}_h \mapsto \mathbf{t}_h := \mathbb{T}(p_h, \mathbf{v}_h)\mathbf{n}_h; \mathbb{T}(p, v) = -p\mathbb{I} + 2\mu\mathbb{D}$

Direct projection

$p_h,\ \mathbf{v}_h \mapsto \int_\Gamma \mathbf{t}_h \cdot \varphi \,\mathrm{d}S := \int_\Gamma \mathbb{T}(p_h, \mathbf{v}_h)\mathbf{n}_h \cdot \varphi \,\mathrm{d}S$

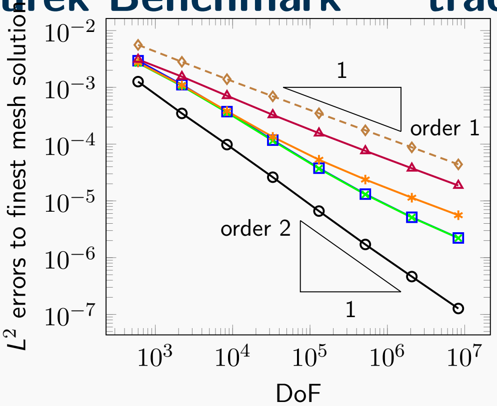
Variational approach

$p_h,\ \mathbf{v}_h \mapsto \int_\Gamma \mathbf{t}_\mathbf{n}^h \cdot \varphi_h \,\mathrm{d}S := \int_\Omega \rho \frac{d\mathbf{v}_h}{dt} \cdot \varphi_h + \int_\Omega \mathbb{T}_h \cdot \nabla\varphi_h - \int_{\partial\Omega\setminus\Gamma} (\mathbb{T}_h \mathbf{n}_h) \cdot \varphi_h \,\mathrm{d}S \qquad \forall \varphi_h \in V^h$

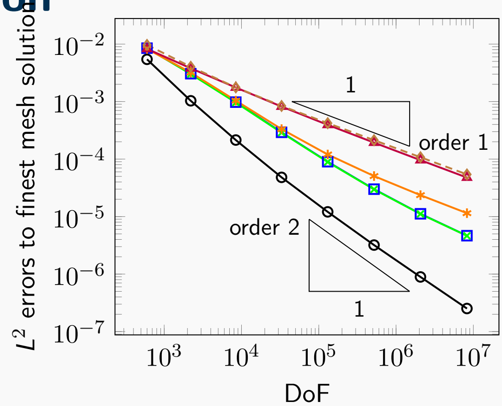
Discrete variational normal derivative ($\Delta u = f, u|_{\partial\Omega} = 0$)

$u_h \mapsto \int_{\partial\Omega} \partial_\mathbf{n}^h u_h\, v_h \,\mathrm{d}S := \int_\Omega \nabla u_h \cdot \nabla v_h \,\mathrm{d}x - \langle f, v_h \rangle \qquad \forall v_h \in V^h$, solve for $\partial_\mathbf{n}^h u \in \operatorname{trace} V^h$

Turek Benchmark — traction



(a) Stokes



(b) Navier-Stokes (original benchmark)