A shadow utility of portfolios efficient with respect to the second order stochastic dominance

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Abstract. We consider diversification-consistent DEA models which are consistent with the second order stochastic dominance (SSD). These models can identify the portfolios which are SSD efficient and suggest the revision of portfolio weights for the inefficient ones. There is also a way how to reconstruct the utility of particular investors based on efficient portfolio which they hold. We apply the above mentioned approaches to industry representative portfolios and discuss the risk aversion of the investors. We focus on the sensitivity with respect to various levels of the risk aversion.

Keywords: Data envelopment analysis, diversification, second order stochastic dominance, risk aversion, shadow utility, sensitivity

JEL Classification: C44
AMS Classification: 90C15

1 Introduction

Data envelopment analysis (DEA), introduced in [13], is nowadays an important class of models which serve to access efficiency of decision making units which consume given set of inputs to produce several outputs. The applications ranges from bank branches up to country regions efficiency, cf. [23]. A special attention has been paid to applications in finance, especially to efficiency of mutual funds and investment opportunities in general. Since the seminar work [26], many papers has been published on applications as well as methodology, see, e.g., [5, 11, 14, 25]. Recently, new class of DEA models with diversification, known also as diversification-consistent DEA (DC DEA), was introduced in [19]. These new models overcame the drawback of the traditional DEA models which does not take into account diversification effect between considered investment opportunities. In other words, if risk measures were considered as the inputs, the traditional DEA model underestimated the risk of the combination of investment opportunities and classified some of them as efficient, even though some improvement in the risk criterion is possible. Since the work [19], several DC DEA classes of models were investigated. Note that previously several attempts can be found in the literature, in particular in [11, 12, 17] which were focused on mean–variance, and mean–variance–skewness efficiency. They also introduced shadow utility functions based on the moment criteria. Paper [6] dealt with DC DEA models based on general deviation measures and investigated the strength of the proposed models as well as inclusion of condition on sparsity of portfolios. In [7], the models were generalized and the analysis was extended to coherent risk measures using the directional distance measures. Bootstrap technique was employed to investigate the empirical properties and stability of the models and resulting scores. The dynamic extension was introduced by [21]. They decomposed the overall efficiency of mutual funds over the whole investment period into efficiencies at individual investment periods taking into account dependence among the periods. Paper [8] studied models with Value at Risk inputs and proposed tractable reformulations. Traditional DEA models were used to approximate the efficient frontier and to assess performance of portfolios by [24]. In [22], two directional distance based diversification super-efficiency models for discriminating efficient funds were proposed. Paper [29] was focused on robustness and integrated parameter uncertainty into diversification-consistent DEA models leading to bi-level problems which were then transformed into equivalent single-level DEA problems. Note that in many cases, for discretely distributed returns and proper choices of risk measures, the authors showed that the proposed models can be formulated as linear programming problems which enables to solve even large instances of the obtained problems to optimality.

An important research topic is relation of DEA efficiency and stochastic dominance efficiency. The efficiency with respect to stochastic dominance is a well established concept in financial mathematics since [15, 16], see also [20]. In DEA literature, we can find several papers which were investigating relations to stochastic dominance efficiency, in particular [25] introduced several models which are consistent with second order stochastic dominance (SSD), whereas [18] proposed equivalent models. In [9], an equivalence between new class of diversification-consistent

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DEA models and stochastic dominance efficiency tests with respect to SSD was shown. This relation was further elaborated by [7]. The equivalences were then generalized to N-order stochastic dominance efficiency tests in [10].

Recently, [2] proposed new approach how to incorporate the risk aversion of a particular investor into the DC DEA framework which is equivalent with SSD efficiency. They derived a shadow utility which renders the DC-DEA/SSD efficient portfolios as optimal for the investor. We will focus on this approach and propose an additional sensitivity analysis with respect to the investor risk aversion. The approach relies on spectral risk measures which were proposed by [1] as a special class of coherent risk measures [4]. Using the proper choice of the risk spectra, we can identify the optimal investment opportunity for any risk-averse investor, see [28].

The paper is organized as follows. Section 2 reviews the DC DEA models and the basic notation of efficiency with respect to the second order stochastic dominance. In Section 3, an approach to risk aversion based on spectral risk measures is summarized. Section 4 provides a numerical study with a special attention to sensitivity with respect to the risk aversion.

Below we will assume that n assets with random rates of return R_i are available and we can use any (nonnegative[1]) combination to compose a portfolio. This leads to the following sets of available investment opportunities, or simply portfolios:

$$\mathcal{X} = \left\{ \sum_{i=1}^{n} x_i R_i : \sum_{i=1}^{n} x_i = 1, \ x_i \geq 0 \right\}. \tag{1}$$

2 Diversification consistent DEA models

First, we review the general formulation of a diversification-consistent DEA model as it was proposed in [7]. It employs J return measures $\mathcal{E}_j$ as the outputs and K coherent risk measures $\mathcal{R}_k$ as the inputs. Coherent risk measures were proposed in [4] as real functionals on $\mathcal{L}_p(\Omega)$ space with finite p-th moment (usually $p \in \{1, 2\}$), which fulfill the following axioms:
(R1)translation equivariance: $\mathcal{R}(X + c) = \mathcal{R}(X) - c$ for all $X \in \mathcal{L}_p(\Omega)$ and constants $c \in \mathbb{R}$,
(R2)positive homogenity: $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for all $X \in \mathcal{L}_p(\Omega)$ and all $\lambda \geq 0$,
(R3)subadditivity: $\mathcal{R}(X_1 + X_2) \leq \mathcal{R}(X_1) + \mathcal{R}(X_2)$ for all $X_1, X_2 \in \mathcal{L}_p(\Omega)$,
(R4)monotonicity: $\mathcal{R}(X_1) \leq \mathcal{R}(X_2)$ when $X_1 \geq X_2$, $X_1, X_2 \in \mathcal{L}_p(\Omega)$.

Note that the axioms (R2) and (R3) imply convexity. We say that $\mathcal{E}$ is a return measure if there exists a coherent risk measure $\mathcal{R}$ such that $\mathcal{E} = -\mathcal{R}$. Since both coherent risk as well as return measures can take positive as well as negative values, the DC DEA models proposed by paper [7] were based on the directional distance measures where, for a benchmark portfolio $X_0 \in \mathcal{X}$, the directions are defined as

$$e_j(X_0) = \max_{X \in \mathcal{X}} \mathcal{E}_j(X) - \mathcal{E}_j(X_0), d_k(X_0) = \mathcal{R}_k(X_0) - \min_{X \in \mathcal{X}} \mathcal{R}_k(X). \tag{2}$$

These directions quantify the maximal possible improvements over the risk and return measures for the benchmark portfolio X_0 to reach the efficient frontier. The frontier corresponds to the strong Pareto–Koopmans efficiency, i.e. we say that X_0 is efficient, if there is not other portfolio $X_1 \in \mathcal{X}$ such that

$$\mathcal{R}_k(X_1) \leq \mathcal{R}_k(X_0), \forall k, \ \mathcal{E}_j(X_1) \geq \mathcal{E}_j(X_0), \forall j,$$

with at least one inequality strict. This efficiency can be then accessed by the following diversification-consistent DEA model based on directional distance measure:

$$\begin{aligned}
\min_{\theta_k, \varphi_j, x_i} \ & \frac{1 - \frac{1}{K} \sum_{k=1}^{K} \theta_k}{1 + \frac{1}{J} \sum_{j=1}^{J} \varphi_j} \\
\text{s.t. } \mathcal{E}_j \left(\sum_{i=1}^{n} R_i x_i \right) \ & \geq \ \mathcal{E}_j(X_0) + \varphi_j \cdot e_j(X_0), \ j = 1, \ldots, J, \\
\mathcal{R}_k \left(\sum_{i=1}^{n} R_i x_i \right) \ & \leq \ \mathcal{R}_k(X_0) - \theta_k \cdot d_k(X_0), \ k = 1, \ldots, K, \\
\sum_{i=1}^{n} x_i \ & = \ 1, \ x_i \geq 0, \ \varphi_j \geq 0, \ \theta_k \geq 0,
\end{aligned} \tag{3}$$

[1] Short-sales are not allowed.

where φ_j denotes the fraction of the improvement of the optimal portfolio $\sum_{i=1}^{j} R_i x_i$ over the maximal possible improvement in return $\mathcal{E}_j$. Similarly, θ_k denotes the fraction of the improvement of the optimal portfolio $\sum_{i=1}^{j} R_i x_i$ over the maximal possible improvement in risk $\mathcal{R}_j$. The objective function quantifies the mean improvement in risks in the numerator and the mean improvements in returns in the denominator. If the optimal is equal to one, then X_0 is identified as efficient, otherwise it is inefficient and the optimal solution (weights x_i) corresponds to an efficient portfolio which can be seen as a projection to an efficient frontier and used to revise the inefficient portfolio. However, the projection need not be in relation with the investor's risk aversion.

Formal definition of the second-order stochastic dominance (SSD) efficiency over $\mathcal{L}_1(\Omega)$ space is based on the twice cumulative probability distribution function of $X \in \mathcal{L}_1(\Omega)$ defined by

$$F_X^{(2)}(t) \quad = \quad \int_{-\infty}^{t} F_X(\eta)\,\mathrm{d}\eta,$$

where $F_X(t) = P(X \leq t)$ is cdf. We say that X dominates $\tilde{X}$ with respect to the second-order stochastic dominance (SSD), $\tilde{X} \preceq_{SSD} X$, if and only if

$$F_X^{(2)}(t) \leq F_{\tilde{X}}^{(2)}(t),\ \forall t \in \mathbb{R}, \tag{4}$$

The relation is strict, i.e. $\tilde{X} \prec_{SSD} X$, if the inequality is strict for at least one $t \in \mathbb{R}$. We say that $X \in \mathcal{X}$ is SSD efficient if there is no other $\tilde{X} \in \mathcal{X}$ for which it holds $X \prec_{SSD} \tilde{X}$. Paper [7] showed that if the distribution of random retusn is discrete with S equiprobable realizations, the inputs in DC DEA model (3) correspond to Conditional Value at Risks (CVaRs, [27]) on levels $1/S, 2/S, \ldots, 1$ and the output is the expected return, then the resulting DC DEA model is equivalent to SSD efficiency tests. In other words, the portfolio is SSD efficient if and only if it is DC DEA efficient.

3 Risk aversion and spectral risk measures

Spectral risk measures (SRM), cf. [1], is a special subclass of the coherent risk measures. They are defined as the weighted quantiles of the random returns

$$M_\phi(X) = -\int_0^1 F_X^{-1}(p)\ \phi(p)\ dp \tag{5}$$

where we consider the quantile function

$$F_X^{-1}(p) = \min\{x :\ F_X(x) \geq p\},\ p \in [0,1], \tag{6}$$

and an admissible risk spectrum which must be:
(A1)positive: for all $I \subseteq [0,1]$ holds

$$\int_I \phi(p)dp \geq 0,$$

(A2)non-increasing: for all $q \in (0,1)$ and $\varepsilon > 0$ such that $[q-\varepsilon, q+\varepsilon] \subset [0,1]$, holds

$$\int_{q-\varepsilon}^{q} \phi(p)dp \geq \int_{q}^{q+\varepsilon} \phi(p)dp,$$

(A3)normalized:

$$\|\phi\| = \int_0^1 \phi(p)dp = 1.$$

Note that Conditional Value at Risk on level α can be obtained as a special case for risk spectrum for the risk spectrum

$$\phi(p) = \frac{1}{1-\alpha}\mathbb{I}\{0 \leq p \leq 1-\alpha\}.$$

In general, investors can identify their risk aversion by choosing the risk spectrum ϕ and derive the admissible empirical risk spectrum using the formula

$$\phi_s = \frac{\phi(s/S)}{\sum_{s=1}^{S} \phi(s/S)}. \tag{7}$$

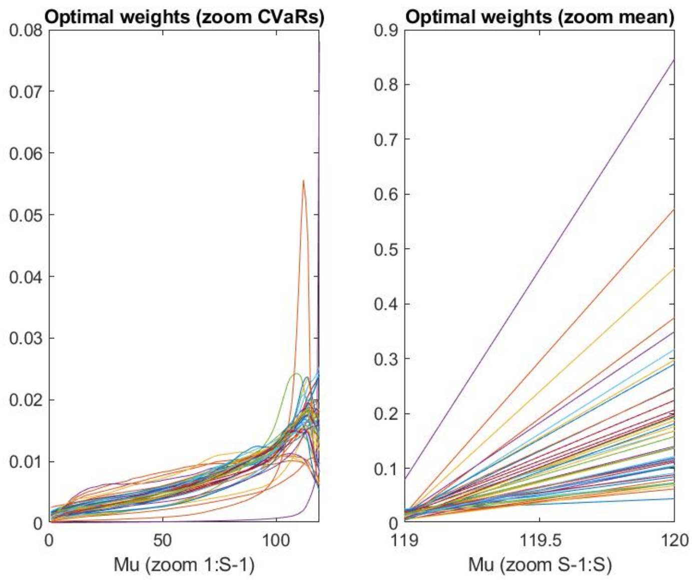

Figure 1 Optimal weights μ_s for efficient projected portfolios

Each empirical spectral risk measure (5) can be then expressed as

$$M_\phi^S(X) = \sum_{s=1}^{S} \mu_s \mathrm{CVaR}_{1-s/S}^S(X) = \mu_S \mathbb{E}(-X) + \sum_{s=1}^{S-1} \mu_s \mathrm{CVaR}_{1-s/S}^S(X), \tag{8}$$

where the weights can be derived from the empirical risk spectrum using the relation

$$\mu_s = s(\phi_s - \phi_{s+1}),\ s = 1, \ldots, S, \tag{9}$$

together with $\phi_{S+1} \equiv 0$. If the investors identify the risk spectrum corresponding to their risk aversion, they can obtain an ideal portfolio by solving the problem (8).

From Proposition 4.1 in [7] we know that the optimal solution of DC DEA model (3) under mean–CVaRs choice of the output–inputs corresponds to the SSD efficient portfolio and using Corollary 3.3 in [2] we can construct the shadow empirical risk spectrum which renders the projection as optimal in SRM minimization problem (8). If $\bar{x}$ are the optimal portfolio weigths for the benchmark portfolio X_0 and $\bar{X} = \sum_{i=1}^{n} R_i \bar{x}_i$ the corresponding random return of the optimal (efficient) portfolio, then weights can be obtained as

$$\begin{aligned} \mu_S &= \frac{1}{e(X_0)} \sum_{s=1}^{S-1} \frac{\mathrm{CVaR}_{s/S}^S(\bar{X}) - \mathrm{CVaR}_{s/S}^S(X_0) + d_s(X_0)}{d_s(X_0)}, \\ \mu_s &= \frac{1}{d_{S-s}(X_0)} \frac{\mathbb{E}(\bar{X}) - \mathbb{E}(X_0) + e(X_0)}{e(X_0)},\ s = 1, \ldots, S-1, \end{aligned} \tag{10}$$

and the shadow empirical risk spectrum as

$$\phi_s = \sum_{t=s}^{S} \frac{\mu_t}{t},\ s = 1, \ldots, S. \tag{11}$$

Figure 1 shows the weights obtained for the projected industry representative portfolios which are analyzed in the numerical study. We can compare the obtained empirical risk spectrum with various theoretical risk spectra which model various levels of investor's risk aversion.

4 Empirical study and sensitivity analysis

In this section, we access efficiency of the industry representative portfolios of US stock market which are listed in the Kenneth French online library. We consider monthly returns between 2012 and 2020. We apply the above

	$k \in \{1,2,3,4\}$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$	$k=12$	$k \in \{13,14,15\}$
Clths	6	6	6	5	3	17	43	42	42	42
Hlth	21	21	21	21	21	18	15	25	26	27
MedEq	5	5	5	4	2	14	41	43	43	43
Chems	16	16	16	16	15	8	27	30	32	32
Txtls	3	3	2	1	5	25	45	45	45	45
BldMt	17	17	17	17	16	9	26	32	31	31
Cnstr	36	36	36	36	36	35	12	15	13	12
FabPr	20	20	20	20	19	16	23	29	28	28
Mach	18	18	18	18	17	10	29	31	30	30
Ships	1	2	3	7	20	40	47	47	47	47
BusSv	7	7	7	6	4	13	40	41	41	41
Comps	25	25	25	25	25	22	18	26	25	23
LabEq	2	1	1	2	9	32	46	46	46	46
Banks	19	19	19	19	18	12	16	28	29	29
Insur	10	10	10	10	8	1	33	37	38	38
RlEst	9	9	9	9	7	5	37	38	39	39
Oil	47	47	47	47	47	47	42	5	5	1

Table 1 Representative portfolios with most changes in the ranking with respect to the risk aversion

introduced approaches, in particular we will investigate the sensitivity of the proposed DC DEA model and its solutions with respect to various risk aversions expressed by exponential risk spectrum:

$$\phi_k(p) = \frac{k \cdot e^{-k \cdot p}}{1 - e^{-k}}, \ k > 0, \tag{12}$$

We will compare the derived shadow risk spectrum and the investor's one. We consider parameters $k \in \{1, 2, \ldots, 15\}$ which cover most of the realistic risk aversions of real investors.

Table 1 contains the industry representative portfolios with most changes in ranking according to the risk aversion and using the distance between the ideal and projected portfolios. We can observe that portfolio Ships is the best according to the risk aversion represented by parameters $k \in \{1, 2, 3, 4\}$, whereas is one the worst for $k \in \{10, \ldots, 15\}$. On the other hand, portfolio Oil, which is the best for $k \in \{13, 14, 15\}$ is very far from the ideal portfolio for $k \in \{1, \ldots, 10\}$. To summarize, we can observe that the ranking is highly dependent of the risk aversion level.

5 Conclusions

In this paper, we have reviewed the diversification-consistent DEA models which are equivalent to the stochastic dominance tests with respect to the second order stochastic dominance. We have proposed a sensitivity analysis of the ranking of considered industry representative portfolios with respect to the various levels of the investor's risk aversion. In particular, we have compared distances between the shadow and the theoretical (empirical) risk spectra showing high dependence of the ranking on the risk aversion parameters. More demanding models are postponed as a topic for future research where they can be solved using the numerical technique proposed in [3].

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