Homework 1 – Linear programming and CVaR

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COMPUTATIONAL ASPECTS OF OPTIMIZATION

Var & CVaR

Let Z be a random variables that represent **loss** with p.d.f. $F_Z(\eta)$, $\alpha \in (0,1)$ (usually $\alpha = 0.95$).

Value at Risk (VaR)

$$\mathrm{VaR}_\alpha(Z) \cong F_Z^{-1}(\alpha)$$

Conditional Value at Risk (CVaR)

$$\mathrm{CVaR}_\alpha(X) \cong \mathbb{E}[Z | Z \geq \mathrm{VaR}_\alpha(Z)]$$

or

$$\mathrm{CVaR}_\alpha(X) \cong \mathbb{E}[Z | Z > \mathrm{VaR}_\alpha(Z)]$$

Exact definitions follow (Rockafellar and Uryasev 2002) ...

Var & CVaR

Value at Risk is defined as a quantile

$$\mathrm{VaR}_\alpha(Z) = \inf\{\eta : P(Z \leq \eta) \geq \alpha\}.$$

For $Z \in \mathcal{L}_1(\Omega)$, **Conditional Value at Risk** (CVaR) is defined as the mean of losses in the α-tail distribution with the distribution function:

$$F_\alpha(\eta) = \begin{cases} \frac{F(\eta)-\alpha}{1-\alpha}, & \text{if } \eta \geq \mathrm{VaR}_\alpha(Z), \\ 0, & \text{otherwise}, \end{cases}$$

where $F(\eta) = P(Z \leq \eta)$. CVaR can be expressed using the following **minimization formula**:

$$\mathrm{CVaR}_\alpha(Z) = \min_{\xi \in \mathbb{R}} \left[\xi + \frac{1}{1-\alpha} \mathbb{E}\left[\max\{Z - \xi, 0\}\right] \right] \tag{1}$$

with the minimum attained at any $(1-\alpha)$-th quantile.

Investment problem with CVaR

Solve a simple investment problem

$$\min_{x_i} \mathrm{CVaR}_\alpha \left(-\sum_{i=1}^{n} x_i R_i \right)$$
$$\text{s.t. } \mathbb{E}\left[\sum_{i=1}^{n} x_i R_i \right] \geq r_0,$$
$$\sum_{i=1}^{n} x_i = 1, \; x_i \geq 0,$$

where we consider n assets with random rate of return R_i. The first constraint ensures minimal expected return r_0, x_i are (nonnegative) portfolio weights which sum to one.

If the distribution of R_i is discrete with realizations r_{is} and probabilities $p_s = 1/S$, then we can use **linear programming** reformulation

$$
\begin{aligned}
\min_{\xi, x_i, u_s} \ & \xi + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} u_s, \\
\text{s.t. } & u_s \geq - \sum_{i=1}^{n} x_i r_{is} - \xi, \ s = 1, \ldots, S, \\
& \sum_{i=1}^{n} x_i \overline{R}_i \geq r_0, \\
& \sum_{i=1}^{n} x_i = 1, \ x_i \geq 0, \\
& \xi \in \mathbb{R}, \ u_s \geq 0,
\end{aligned}
$$

where $\overline{R}_i = 1/S \sum_{s=1}^{S} r_{is}$.

Numerical experiment

1. Use at least 6 assets and 100 return realizations.
2. Run the problem for different 11 values $r_0 \in \{\min_i \overline{R}_i, \ldots, \max_i \overline{R}_i\}$.
3. Plot the optimal values CVaR_α against the corresponding values of r_0.