Dynamic programming in discrete time – examples

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Computational Aspects of Optimization

Dynamic programming

Finite T or infinite ∞ time horizon

$$\max_{A_t, c_t} \sum_{t=1}^{T \vee \infty} \left(\frac{1}{1+i} \right)^{t-1} u(c_t)$$
$$\text{s.t.}$$
$$A_t = (1+r)A_{t-1} + Y_t - c_t. \tag{1}$$

- u – **utility function**
- A_t – **state variables** representing total amount of resources available to the consumer.
- c_t – **control variables** maximizing the consumer's utility. It affects the resources available in the next period.
- Y_t – **exogenous income**
- $1/(1+i)$ – discount factor, r – exogenous interest rate

Dynamic programming

If we assume that there is a finite terminal period T:

$$\begin{aligned}
V_1(A_0) &= \max_{A_t, c_t} \sum_{t=1}^{T} \left(\frac{1}{1+i}\right)^{t-1} u(c_t) \\
&= \max_{A_t, c_t} u(c_1) + \frac{1}{1+i} u(c_2) + \cdots + \left(\frac{1}{1+i}\right)^{T-1} u(c_T) \\
&= \max_{A_t, c_t} u(c_1) + \frac{1}{1+i} \left[\sum_{t=2}^{T} \left(\frac{1}{1+i}\right)^{t-2} u(c_t)\right] \\
& \text{s.t.} \\
& A_t = (1+r)A_{t-1} + Y_t - c_t.
\end{aligned}$$

Dynamic programming

We rewrite the maximization problem recursively and obtain the **Bellman equation**

$$V_t(A_{t-1}) = \max_{A_t, c_t} u(c_t) + \frac{1}{1+i} V_{t+1}(A_t),$$

where $A_t = (1+r)A_{t-1} + Y_t - c_t$.

Moreover, since u does not depend on the time period, we can write

$$\begin{aligned} V(A_{t-1}) &= \max_{A_t, c_t} u(c_t) + \frac{1}{1+i} V(A_t), \\ &= \max_{c_t} u(c_t) + \frac{1}{1+i} V\Big((1+r)A_{t-1} + Y_t - c_t\Big), \end{aligned}$$

with $V_{T+1}(A_T) = V(A_T) \equiv 0$.

Dynamic programming

First order optimality conditions

$$\begin{aligned} \frac{\partial V(A_{t-1})}{\partial c_t} &= 0, \\ \frac{\partial V(A_{t-1})}{\partial A_{t-1}} &= 0. \end{aligned}$$

In particular,

$$\begin{aligned} \frac{\partial V(A_{t-1})}{\partial c_t} &= u'(c_t) + \frac{1}{1+i} V'(A_t) \frac{\partial A_t}{\partial c_t}, \\ \frac{\partial V(A_{t-1})}{\partial A_{t-1}} &= \frac{1}{1+i} V'(A_t) \frac{\partial A_t}{\partial A_{t-1}}, \end{aligned}$$

where using $A_t = (1+r)A_{t-1} + Y_t - c_t$ we have

$$\frac{\partial A_t}{\partial c_t} = -1, \quad \frac{\partial A_t}{\partial A_{t-1}} = 1 + r.$$

Cake eating problem:

- $u(c) = 2c^{1/2}$,
- $\Pi_0 = 1$, $\Pi_T = 0$,
- $\Pi_t = \Pi_{t-1} - c_t \ \ldots$

Example: Cake eating problem

Bellman equation

$$V(\Pi_{t-1}) = \max_{c_t} \; u(c_t) + \frac{1}{1+i} V(\Pi_t),$$
$$\text{s.t. } \Pi_t = \Pi_{t-1} - c_t.$$

Optimality conditions

$$\frac{\partial V(\Pi_{t-1})}{\partial c_t} = u'(c_t) + \frac{1}{1+i} V'(\Pi_t) \frac{\partial \Pi_t}{\partial c_t} = 0,$$
$$\frac{\partial V(\Pi_{t-1})}{\partial \Pi_{t-1}} = \frac{1}{1+i} V'(\Pi_t) \frac{\partial \Pi_t}{\partial \Pi_{t-1}} = 0,$$

From $\Pi_t = \Pi_{t-1} - c_t$

$$\frac{\partial \Pi_t}{\partial c_t} = -1, \; \frac{\partial \Pi_t}{\partial \Pi_{t-1}} = 1. \tag{2}$$

Example: Cake eating problem

Putting them together, we obtain

$$\frac{\partial V(\Pi_{t-1})}{\partial c_t} = u'(c_t) - \frac{1}{1+i} V'(\Pi_t) = 0, \quad (3)$$

$$\frac{\partial V(\Pi_{t-1})}{\partial \Pi_{t-1}} = \frac{1}{1+i} V'(\Pi_t) = 0, \quad (4)$$

Taking (3) for $t-1$

$$u'(c_{t-1}) - \frac{1}{1+i} V'(\Pi_{t-1}) = 0, \quad (5)$$

and plugging it into (4), we have

$$u'(c_{t-1}) = \frac{1}{1+i} u'(c_t),$$

which represents the **optimal path of the cake consumption**.

Example: Cake eating problem

For $u(c) = 2c^{1/2}$, we have

$$\begin{aligned} u'(c_{t-1}) &= \frac{1}{1+i} u'(c_t), \\ (c_{t-1})^{-1/2} &= \frac{1}{1+i} (c_t)^{-1/2}, \\ c_t &= \left(\frac{1}{1+i}\right)^2 c_{t-1}, \end{aligned}$$

with initial and terminal conditions $\Pi_0 = 1$, $\Pi_T = 0$. If we denote $\beta = 1/(1+i)^2$, we obtain

$$c_t = \beta c_{t-1} = \beta^{t-1} c_1.$$

Using

$$c_t = \beta c_{t-1} = \beta^{t-1} c_1.$$

and

$$\Pi_0 - c_1 - c_2 - \ldots - c_T = \Pi_T = 0,$$

we have

$$(1 - \beta - \ldots - \beta^{T-1}) c_1 = \Pi_0,$$

and finally **optimal consumption**

$$\begin{aligned} \hat{c}_1 &= \frac{1-\beta}{1-\beta^T} \Pi_0, \\ \hat{c}_t &= \beta \hat{c}_{t-1} = \beta^{t-1} \hat{c}_1. \end{aligned}$$

Literature

- M. C. Sunny Wong: **Dynamic Optimization: An Introduction**, Lecture Notes – University of Houston, 2013.