Introduction into Vehicle Routing Problems and other basic mixed-integer problems

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Computational Aspects of Optimization

Totally unimodular matrices

Definition

A matrix A is totally unimodular (TU) iff every square submatrix of A has determinant $+1$, -1, or 0.

The linear program has an integral optimal solution for all integer r.h.s. b if and only if A is TU.

Totally unimodular matrices

A set of **sufficient conditions**:

- $a_{ij} \in \{-1, 0, 1\}$ for all i, j
- Each column contains at most two nonzero coefficients, i.e. $\sum_{i=1}^{m} |a_{ij}| \leq 2$,
- There exists a partitioning $M_1 \cap M_2 = \emptyset$ of the rows $1, \dots, m$ such that each column j containing two nonzero coefficients satisfies

$$\sum_{i \in M_1} a_{ij} = \sum_{i \in M_2} a_{ij}.$$

If A is TU, then A^T and $(A|I)$ are TU.

Minimum cost network flow problem

- $G = (V, A)$ – graph with vertices V and (oriented) arcs A
- h_{ij} – arc capacity
- c_{ij} – flow cost
- b_i – demand, ASS. $\sum_i b_i = 0$
- $V^+(i) = \{k : (i,k) \in A\}$ – successors of i
- $V^-(i) = \{k : (k,i) \in A\}$ – predecessors of i

$$\begin{aligned}
\min_{x_{ij}} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{k\in V^+(i)} x_{ik} - \sum_{k\in V^-(i)} x_{ki} = b_i, \; i \in V, \\
& 0 \le x_{ij} \le h_{ij}, \; (i,j) \in A.
\end{aligned}$$

Wolsey (1998), Ex. 3.1 ($M_1 = \{1, \ldots, m\}, M_2 = \emptyset$)

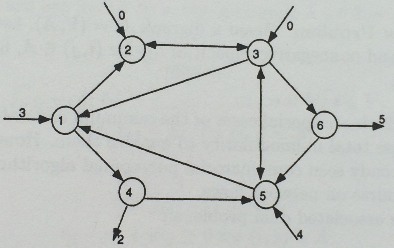


Fig. 3.1 Digraph for minimum cost network flow

equations:

x_{12}	x_{14}	x_{23}	x_{31}	x_{32}	x_{35}	x_{36}	x_{45}	x_{51}	x_{53}	x_{65}		
1	1	0	−1	0	0	0	0	−1	0	0	=	3
−1	0	1	0	−1	0	0	0	0	0	0	=	0
0	0	−1	1	1	1	1	0	0	−1	0	=	0
0	−1	0	0	0	0	0	1	0	0	0	=	−2
0	0	0	0	0	−1	0	−1	1	1	−1	=	4
0	0	0	0	0	0	−1	0	0	0	1	=	−5

Special cases

- The shortest path problem
- The transportation problem

Traveling salesman problem

- n town and in one of them there is a traveling salesman.
- Traveling salesman must visit all towns and return back.
- For each pair of towns he/she knows the traveling costs and he is looking for the cheapest route.

$=$ Finding a Hamilton cycle in a graph with edge prices.

Assignment problem

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{1}$$

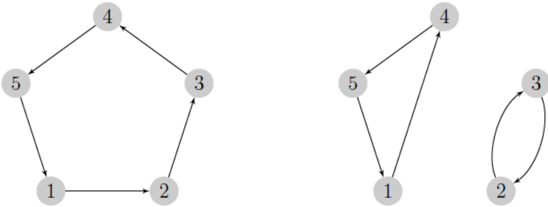
$$\sum_{i=1}^{n} x_{ij} = 1, \; j = 1, \dots, n, \tag{2}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \; i = 1, \dots, n, \tag{3}$$

$$x_{ij} \in \{0, 1\}. \tag{4}$$

We minimize the traveling costs, we arrive to j from exactly one i, we leave i to exactly one j.

Example – 5 towns – cycle and subcycles (subroute)



Kafka (2013)

Subroute elimination conditions I

- $x_{ii} = 0$, $c_{ii} = \infty$
- $x_{ij} + x_{ji} \leq 1$
- $x_{ij} + x_{jk} + x_{ki} \leq 2$
- $\dots$
- $\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1$, $S \subseteq \{1, \dots, n\}$, $2 \leq |S| \leq n - 1$

Approximately 2^n inequalities, it is possible to reduce to $|S| \leq \lceil n/2 \rceil$.

Subroute elimination conditions II

$$u_i - u_j + n x_{ij} \leq n - 1, i, j = 2, \ldots, n$$

Eliminate subroutes: There is at least one route which does not go through vertex 1, denote this route by C and the number of edges by $|E(C)|$. If we sum these inequalities over all edges $\{i, j\}$, which are in C, i.e. the corresponding variables $x_{ij} = 1$, we obtain

$$n|E(C)| \leq (n-1)|E(C)|, \tag{5}$$

which is a contradiction.

Subroute elimination conditions III

$$u_i - u_j + nx_{ij} \leq n - 1, i, j = 2, \ldots, n$$

Hamilton cycle is feasible: let the vertices be ordered as $v_1 = 1$, v_2, ..., v_n. We set $u_i = l$, if $v_l = i$, i.e. u_i represent the order. For each edge of the cycle $\{i, j\}$ it holds $u_i - u_j = -1$, i.e.

$$u_i - u_j + nx_{ij} = -1 + n \leq n - 1. \quad (6)$$

For edges, which are not in the cycle, the inequality holds too: $u_i - u_j \leq n - 1$ a $x_{ij} = 0$.

Traveling Salesman Problem with Time Windows

- t_i – time when customer i is visited
- T_{ij} – time necessary to reach j from i
- l_i, u_i – lower and upper bound (time window) for visiting customer i
- M – large constant

$$\min_{x_{ij}, t_i} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{7}$$

$$\sum_{i=1}^{n} x_{ij} = 1, \; j = 1, \dots, n, \tag{8}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \; i = 1, \dots, n, \tag{9}$$

$$t_i + T_{ij} - t_j \leq M(1 - x_{ij}) \; i, j = 1, \dots, n, \tag{10}$$

$$l_i \leq t_i \leq u_i, \; i = 1, \dots, n, \tag{11}$$

$$x_{ij} \in \{0, 1\}.$$

Capacitated Vehicle Routing Problem

Parameters

- n – number of customers
- 0 – depo (starting and finishing point of each vehicle)
- K – number of vehicles (homogeneous)
- $d_j \geq 0$ – customer demand, for depo $d_0 = 0$
- $Q > 0$ – vehicle capacity ($KQ \geq \sum_{j=1}^{n} d_j$)
- c_{ij} – transportation costs from i to j (usually $c_{ii} = 0$)

Decision variables

- x_{ij} – equal to 1, if j follows after i on the route, 0 otherwise
- u_j – upper bound on transported amount after visiting customer j

Capacitated Vehicle Routing Problem

$$\min_{x_{ij}, u_i} \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} x_{ij} \tag{12}$$

$$\sum_{i=0}^{n} x_{ij} = 1, \; j = 1, \dots, n, \tag{13}$$

$$\sum_{j=0}^{n} x_{ij} = 1, \; i = 1, \dots, n, \tag{14}$$

$$\sum_{i=1}^{n} x_{i0} = K, \tag{15}$$

$$\sum_{j=1}^{n} x_{0j} = K, \tag{16}$$

$$u_i - u_j + d_j \leq Q(1 - x_{ij}) \; i, j = 1, \dots, n, \tag{17}$$

$$d_i \leq u_i \leq Q, \; i = 1, \dots, n, \tag{18}$$

$$x_{ij} \in \{0, 1\}.$$

Capacitated Vehicle Routing Problem

(12) minimization of transportation costs

(13) exactly one vehicle arrives to customer j

(14) exactly one vehicle leaves customer i

(15) exactly K vehicles return to depot 0

(16) exactly K vehicles leave depot 0

(17) balance conditions of transported amount (subroute elimination conditions)

(18) bounds on the vehicle capacity

(All vehicles are employed.)

Basic heuristics for VRP

Insertion heuristic:

- Start with empty routes.
- FOR all customers DO: Insert the customer to the place in a route where it causes the lowest increase of the traveled distance.

Clustering:

- Cluster the customers according to their geographic positions ("angles").
- Solve[1] the traveling salesman problem in each cluster.

Possible difficulties: time windows, vehicle capacities, ...

[1] ..exactly, if the clusters are not large.

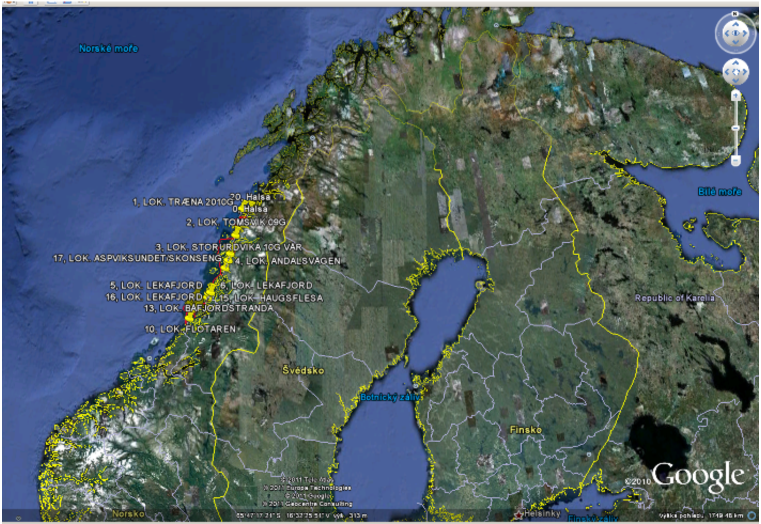
Tabu search for VRP

For a given number of iteration, run the following steps:

- Find the best solution in a *neighborhood* of the current solution. Such solution can be worse than the current one or even infeasible (use penalty function).
- Forbid moving back for a random number of steps by actualizing the **tabu list**.
- Remember the best solution.

The tabu search algorithm enables moving from local solutions (compared with a simple “hill climbing alg.”).

Norway . . .


Norské moře
1, LOK. TRÆNA 2010G
20. Halsa
0. Halsa
2. LOK. TOMSVIK 09G
3. LOK. STORURDVIKA 10G VÅR
17. LOK. ASPVIKSUNDET/SKONSENG
4. LOK. ANDALSVAGEN
5. LOK. LEKAFJORD
6. LOK. LEKAFJORD
16. LOK. LEKAFJORD
15. LOK. HAUGSFLESA
13. LOK. BAFJORDSTRANDA
10. LOK. FLOTAREN
Bílé moře
Republic of Karelia
Švédsko
Botnický záliv
Finsko
Google
Norsko
Helsinky

Rich Vehicle Routing Problems

- **Goal** – maximization of the ship *filling rate* (operational planning), optimization of fleet composition, i.e. number and capacity of the ships (strategic planning)
- **Rich Vehicle Routing Problem**
  - time windows
  - heterogeneous fleet (vehicles with different capacities and speed)
  - several depots with inter-depot trips
  - several routes during the planning horizon
  - *non-Euclidean distances* (fjords)
- Mixed-integer programming :-(, constructive heuristics for getting an initial feasible solution and tabu search

- M. Branda, K. Haugen, J. Novotný, A. Olstad, **Downstream logistics optimization at EWOS Norway**. Research report.

Rich Vehicle Routing Problems

Our approach

- Mathematical formulation
- GAMS implementation
- Heuristic (insertion, tabu search) implementation
- Decision Support System (DSS)

Facility Location Problem

- i warehouses (facilities), j customers
- x_{ij} – sent quantity
- y_i – a warehouse is built
- c_{ij} – unit supplying costs
- f_i – fixed costs
- K_i – warehouse capacity
- D_j – demand

$$
\begin{aligned}
\min_{x_{ij}, y_i} \; & \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} + \sum_{i} f_i y_i \\
\text{s.t.} \; & \sum_{j=1}^{m} x_{ij} \leq K_i y_i, \;\; i = 1, \ldots, n, \\
& \sum_{i=1}^{n} x_{ij} = D_j, \;\; j = 1, \ldots, m, \\
& x_{ij} \geq 0, \;\; y_i \in \{0, 1\}.
\end{aligned}
$$

Scheduling to Minimize the Makespan

- i machines, j jobs,
- y – machine makespan,
- x_{ij} – assignment variable
- t_{ij} – time necessary to process job j on machine i,

$$\begin{aligned} \min_{x_{ij},y} \ & y \\ \text{s.t.} \ & \sum_{i=1}^{m} x_{ij} = 1, \ j = 1, \ldots, n, \\ & \sum_{j=1}^{n} x_{ij} t_{ij} \leq y, \ \ i = 1, \ldots, m. \end{aligned} \tag{19}$$

Lot Sizing Problem

Uncapacitated single item LSP

- x_t – production at period t
- y_t – on/off decision at period t
- s_t – inventory at the end of period t ($s_0 \geq 0$ fixed)
- D_t – (predicted) *expected* demand at period t
- p_t – unit production costs at period t
- f_t – setup cost at period t
- h_t – inventory cost at period t
- M – large constant

$$\begin{aligned} \min_{x_t, y_t, s_t} \ & \sum_{t=1}^{T} (p_t x_t + f_t y_t + h_t s_t) \\ \text{s.t. } & s_{t-1} + x_t - D_t = s_t, \ t = 1, \ldots, T, \\ & x_t \leq M y_t, \\ & x_t, s_t \geq 0, \ y_t \in \{0, 1\}. \end{aligned} \tag{20}$$

ASS. Wagner-Whitin costs $p_{t+1} \leq p_t + h_t$.

Lot Sizing Problem
Capacitated single item LSP

- x_t – production at period t
- y_t – on/off decision at period t
- s_t – inventory at the end of period t ($s_0 \geq 0$ fixed)
- D_t – (predicted) *expected* demand at period t
- p_t – unit production costs at period t
- f_t – setup cost at period t
- h_t – inventory cost at period t
- C_t – production capacity at period t

$$\begin{aligned} \min_{x_t, y_t, s_t} \ & \sum_{t=1}^{T} (p_t x_t + f_t y_t + h_t s_t) \\ \text{s.t. } & s_{t-1} + x_t - D_t = s_t,\ t = 1, \dots, T, \\ & x_t \leq C_t y_t, \\ & x_t, s_t \geq 0,\ y_t \in \{0, 1\}. \end{aligned} \tag{21}$$

ASS. Wagner-Whitin costs $p_{t+1} \leq p_t + h_t$.

Unit Commitment Problem

- y_{it} – on/off decision for unit i at period t
- x_{it} – production level for unit i at period t
- D_t – (predicted) *expected* demand at period t
- p_i^{min}, p_i^{max} – minimal/maximal production capacity of unit i
- c_{it} – (fixed) start-up costs
- f_{it} – variable production costs

$$\begin{aligned} \min_{x_{it}, y_{it}} \ & \sum_{i=1}^{n} \sum_{t=1}^{T} (c_{it} x_{it} + f_{it} y_{it}) \\ \text{s.t.} \ & \sum_{i=1}^{n} x_{it} \geq D_t, \ t = 1, \dots, T, \\ & p_i^{min} y_{it} \leq x_{it} \leq p_i^{max} y_{it}, \\ & x_{it} \geq 0, \ y_{it} \in \{0, 1\}. \end{aligned} \tag{22}$$

Literature

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- P. Toth, D. Vigo (2002). **The vehicle routing problem**, SIAM, Philadelphia.
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