Introduction to Integer Linear Programming

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Computational Aspects of Optimization

Knapsack problem

Values $a_1 = 4$, $a_2 = 6$, $a_3 = 7$, costs $c_1 = 4$, $c_2 = 5$, $c_3 = 11$, budget $b = 10$:

$$\begin{aligned} \max \ & \sum_{i=1}^{3} c_i x_i \\ \text{s.t.} \ & \sum_{i=1}^{3} a_i x_i \leq 10, \\ & x_i \in \{0,1\}. \end{aligned}$$

Consider $=$ instead of $\leq$, $0 \leq x_i \leq 1$ and rounding instead of $x_i \in \{0,1\}$, heuristic (ratio c_i/a_i) ...

Why is integrality so important?

Real (mixed-)integer programming problems (not always linear)

- **Portfolio optimization** – integer number of assets, fixed transaction costs
- **Scheduling** – integer (binary) decision variables to assign a job to a machine
- **Vehicle Routing Problems** (VRP) – binary decision variables which identify a successor of a node on the route
- ...

In general – modelling of **logical relations**, e.g.

- at least two constraints from three are fulfilled,
- if we buy this asset than the fixed transaction costs increase,
- ...

Facility Location Problem

- i warehouses (facilities, branches), j customers,
- x_{ij} – sent (delivered, served) quantity,
- y_i – a warehouse is built,
- c_{ij} – unit supplying costs,
- f_i – fixed costs of building the warehouse,
- K_i – warehouse capacity,
- D_j – demand.

$$\begin{aligned} \min_{x_{ij}, y_i} \ & \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} + \sum_i f_i y_i \\ \text{s.t.} \ & \sum_{j=1}^{m} x_{ij} \leq K_i y_i, \ i = 1, \dots, n, \\ & \sum_{i=1}^{n} x_{ij} = D_j, \ j = 1, \dots, m, \\ & x_{ij} \geq 0, \ y_i \in \{0,1\}. \end{aligned}$$

Scheduling to Minimize the Makespan

- i machines, j jobs,
- y – machine makespan,
- x_{ij} – assignment variable,
- t_{ij} – time necessary to process job j on machine i.

$$\begin{aligned} \min_{x_{ij},y}\ & y \\ \text{s.t.}\ & \sum_{i=1}^{m} x_{ij} = 1,\ j = 1,\dots,n, \\ & \sum_{j=1}^{n} t_{ij}x_{ij} \le y,\ i = 1,\dots,m, \\ & x_{ij} \in \{0,1\},\ y \ge 0. \end{aligned} \tag{1}$$

Lot Sizing Problem
Uncapacitated single item LSP

- x_t – production at period t,
- y_t – on/off decision at period t,
- s_t – inventory at the end of period t ($s_0 \ge 0$ fixed),
- D_t – (predicted) *expected* demand at period t,
- p_t – unit production costs at period t,
- f_t – setup costs at period t,
- h_t – inventory costs at period t,
- M – a large constant.

$$\begin{aligned} \min_{x_t,y_t,s_t}\ & \sum_{t=1}^{T}(p_t x_t + f_t y_t + h_t s_t) \\ \text{s.t.}\ & s_{t-1} + x_t - D_t = s_t,\ t = 1,\dots,T, \\ & x_t \le M y_t, \\ & x_t, s_t \ge 0,\ y_t \in \{0,1\}. \end{aligned} \tag{2}$$

ASS. Wagner-Whitin costs $p_{t+1} \le p_t + h_t$.

Lot Sizing Problem
Capacitated single item LSP

- x_t – production at period t,
- y_t – on/off decision at period t,
- s_t – inventory at the end of period t ($s_0 \ge 0$ fixed),
- D_t – (predicted) *expected* demand at period t.
- p_t – unit production costs at period t,
- f_t – setup costs at period t,
- h_t – inventory costs at period t,
- C_t – production capacity at period t.

$$\begin{aligned} \min_{x_t,y_t,s_t}\ & \sum_{t=1}^{T}(p_t x_t + f_t y_t + h_t s_t) \\ \text{s.t.}\ & s_{t-1} + x_t - D_t = s_t,\ t = 1,\dots,T, \\ & x_t \le C_t y_t, \\ & x_t, s_t \ge 0,\ y_t \in \{0,1\}. \end{aligned} \tag{3}$$

ASS. Wagner-Whitin costs $p_{t+1} \le p_t + h_t$.

Unit Commitment Problem

- $i = 1,\dots,n$ units (power plants), $t = 1,\dots,T$ periods,
- y_{it} – on/off decision for unit i at period t,
- x_{it} – production level for unit i at period t,
- D_t – (predicted) *expected* demand at period t,
- p_i^{min}, p_i^{max} – minimal/maximal production capacity of unit i,
- c_{it} – variable production costs,
- f_{it} – (fixed) start-up costs.

$$\begin{aligned} \min_{x_{it},y_{it}}\ & \sum_{i=1}^{n}\sum_{t=1}^{T}(c_{it}x_{it} + f_{it}y_{it}) \\ \text{s.t.}\ & \sum_{i=1}^{n} x_{it} \ge D_t,\ t = 1,\dots,T, \\ & p_i^{min} y_{it} \le x_{it} \le p_i^{max} y_{it}, \\ & x_{it} \ge 0,\ y_{it} \in \{0,1\}. \end{aligned} \tag{4}$$

Sparse l_1 regression

- Y_i – dependent variable $i = 1, \dots, n$,
- X_{ij} – explanatory (independent) variables $j = 1, \dots, m$,
- β_j – coefficients.

$$\min_{\beta_j} \sum_{i=1}^{n} \left| Y_i - \sum_{j=1}^{m} X_{ij}\beta_j \right| \quad \text{s.t. at most } \kappa < m \text{ coefficients are nonzero.} \tag{5}$$

MILP reformulation

$$\begin{aligned} \min_{\beta,u,z} \; & \sum_{i=1}^{n} u_i^+ + u_i^- \\ \text{s.t. } & u_i^+ - u_i^- = Y_i - \sum_{j=1}^{m} X_{ij}\beta_j, \\ & -Mz_j \le \beta_j \le Mz_j, \\ & \sum_{j=1}^{m} z_j \le \kappa, \; u_i^+, u_i^- \ge 0, \; z_j \in \{0,1\}. \end{aligned} \tag{6}$$

Chance constrained problems – single random constraint

Let $f, g(\cdot,\xi) : \mathbb{R}^n \to \mathbb{R}$ be real functions, $X \subseteq \mathbb{R}^n$, ξ be a real random vector, $\varepsilon \in (0,1)$ small:

$$\begin{aligned} & \min_{x \in X} f(x) \\ & \text{s.t.} \quad P\left(g(x,\xi) \le 0\right) \ge 1 - \varepsilon. \end{aligned}$$

INTERPRETATION: for a given $x \in X$, the probability of ξ for which the random constraint is fulfilled must be at least $1 - \varepsilon$:

$$P\left(g(x,\xi) \le 0\right) = P\left(\{\xi : \; g(x,\xi) \le 0\}\right).$$

Chance constrained problems – single random constraint

Let ξ have a finite discrete distribution with realizations $\xi^1, \dots, \xi^S$ and probabilities $p_s > 0$, $\sum_{s=1}^{S} p_s = 1$:

$$\begin{aligned} & \min_{x,y} f(x) \\ & \text{s.t.} \\ \sum_{s=1}^{S} p_s y_s & \ge 1 - \varepsilon, \\ g(x,\xi_s) & \le M(1 - y_s), \; s = 1, \dots, S \\ y_s & \in \{0,1\}, \; s = 1, \dots, S, \\ x & \in X, \end{aligned} \tag{7}$$

where $M \ge \max_{s=1,\dots,S} \sup_{x \in X} g(x,\xi_s)$.

Example: Value at Risk (VaR).

Integer linear programming

$$\min c^T x \tag{8}$$
$$Ax \ge b, \tag{9}$$
$$x \in \mathbb{Z}_+^n. \tag{10}$$

Assumption: all coefficients are integer (rational before multiplying by a proper constant).

Set of feasible solution and its relaxation

$$S = \{x \in \mathbb{Z}_+^n : \; Ax \ge b\}, \tag{11}$$
$$P = \{x \in \mathbb{R}_+^n : \; Ax \ge b\} \tag{12}$$

Obviously $S \subseteq P$. Not so trivial that $S \subseteq \mathrm{conv}(S) \subseteq P$.

ILP – irrational data

Škoda (2010):

$$\begin{aligned} \max\ & \sqrt{2}x - y \\ \text{s.t. } & \sqrt{2}x - y \le 0, \\ & x \ge 1, \\ & x, y \in \mathbb{N}. \end{aligned} \tag{13}$$

The objective value is bounded (from above), but there is no optimal solution.

For any feasible solution with the objective value $z = \sqrt{2}x^* - \lceil \sqrt{2}x^* \rceil$ we can construct a solution with a higher objective value...

ILP – irrational data

Let $z = \sqrt{2}x^* - \lceil \sqrt{2}x^* \rceil$ be the optimal solution. Since $-1 < z < 0$, we can find $k \in \mathbb{N}$ such that $kz < -1$ and $(k-1)z > -1$. By setting $\epsilon = -1 - kz$ we get that $-1 < z < -\epsilon = 1 + kz < 0$. Then

$$\begin{aligned} & \sqrt{2}kx^* - \left\lceil \sqrt{2}kx^* \right\rceil \\ & = kz + k\left\lceil \sqrt{2}x^* \right\rceil - \left\lceil \sqrt{2}kx^* \right\rceil \\ & = -1 - \epsilon + k\left\lceil \sqrt{2}x^* \right\rceil - \left\lceil \sqrt{2}kx^* \right\rceil \\ & = k\left\lceil \sqrt{2}x^* \right\rceil - 1 - \epsilon - \left\lceil k\left\lceil \sqrt{2}x^* \right\rceil - 1 - \epsilon \right\rceil \\ & = -\epsilon > z. \end{aligned} \tag{14}$$

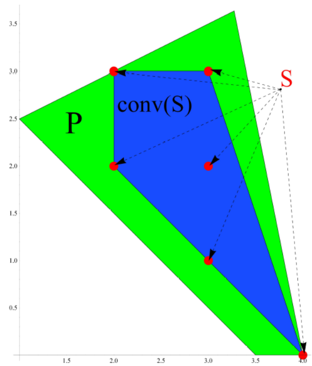
($k\lceil \sqrt{2}x^* \rceil - 1$ is integral)
Thus, we have obtained a solution with a higher objective value which is a contradiction.

Example

Consider set S given by

$$\begin{aligned} 7x_1 + 2x_2 &\ \ge\ 5, \\ 7x_1 + x_2 &\ \le\ 28, \\ -4x_1 + 14x_2 &\ \le\ 35, \\ x_1, x_2 &\ \in\ \mathbb{Z}_+. \end{aligned}$$

Set of feasible solutions, its relaxation and convex envelope



Škoda (2010)

Integer linear programming problem

Problem

$$\min c^T x : \; x \in S. \tag{15}$$

is equivalent to

$$\min c^T x : \; x \in \mathrm{conv}(S). \tag{16}$$

$\mathrm{conv}(S)$ is very difficult to construct – many constraints ("strong cuts") are necessary (there are some important exceptions).

LP-relaxation:

$$\min c^T x : \; x \in P. \tag{17}$$

Mixed-integer linear programming

Often both integer and continuous decision variables appear:

$$\begin{aligned} \min\ & c^T x + d^T y \\ \text{s.t. } & Ax + By \geq b \\ & x \in \mathbb{Z}^n_+, \; y \in \mathbb{R}^{n'}_+. \end{aligned}$$

(WE DO NOT CONSIDER IN INTRODUCTION)

Basic algorithms

We consider:

- **Cutting Plane Method**
- **Branch-and-Bound**

There are methods which combine the previous alg., e.g. **Branch-and-Cut** (add cuts to reduce the problem for B&B).

Cutting plane method – Gomory cuts

1. Solve LP-relaxation using (primal or dual) SIMPLEX algorithm.
   - If the solution is integral – END, we have found an optimal solution,
   - otherwise continue with the next step.
2. Add a **Gomory cut** (...) and solve the resulting problem using DUAL SIMPLEX alg.

Example

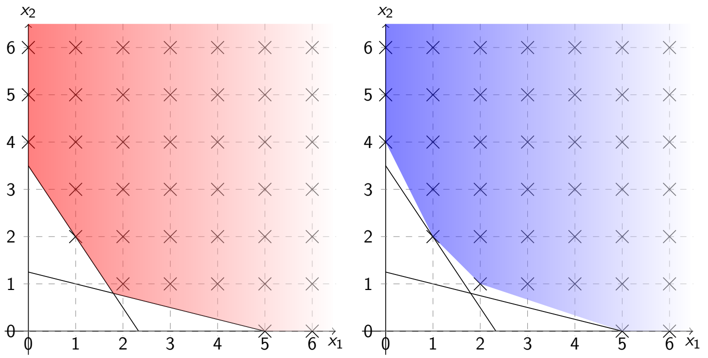
$$\min 4x_1 + 5x_2 \quad (18)$$

$$x_1 + 4x_2 \geq 5, \quad (19)$$

$$3x_1 + 2x_2 \geq 7, \quad (20)$$

$$x_1, x_2 \in \mathbb{Z}^n_+. \quad (21)$$

Dual simplex for LP-relaxation ...



After two iterations of the dual SIMPLEX algorithm ...

			4	5	0	0
			x_1	x_2	x_3	x_4
5	x_2	8/10	0	1	-3/10	1/10
4	x_1	18/10	1	0	2/10	-4/10
		112/10	0	0	-7/10	-11/10

Gomory cuts

There is a row in simplex table, which corresponds to a **non-integral solution** x_i in the form:

$$x_i + \sum_{j \in N} w_{ij} x_j = d_i, \quad (22)$$

where N denotes the set of non-basic variables; d_i is non-integral. We denote

$$w_{ij} = \lfloor w_{ij} \rfloor + f_{ij}, \quad (23)$$

$$d_i = \lfloor d_i \rfloor + f_i, \quad (24)$$

i.e. $0 \leq f_{ij}, f_i < 1$.

$$\sum_{j \in N} f_{ij} x_j \geq f_i, \quad (25)$$

or rather $-\sum_{j \in N} f_{ij} x_j + s = -f_i$, $s \geq 0$.

Gomory cuts

General properties of cuts (including Gomory ones):

- Property 1: Current (non-integral) solution becomes infeasible (it is cut).
- Property 2: No feasible integral solution becomes infeasible (it is not cut).

Gomory cuts – property 1

We express the constraints in the form

$$x_i + \sum_{j\in N} (\lfloor w_{ij} \rfloor + f_{ij}) x_j = \lfloor d_i \rfloor + f_i, \qquad (26)$$

$$x_i + \sum_{j\in N} \lfloor w_{ij} \rfloor x_j - \lfloor d_i \rfloor = f_i - \sum_{j\in N} f_{ij} x_j. \qquad (27)$$

Current solution $x_j^* = 0$ for $j \in N$ and $x_i^* = d_i$ is non-integral, i.e. $0 < x_i^* - \lfloor d_i \rfloor < 1$, thus

$$0 < x_i^* - \lfloor d_i \rfloor = f_i - \sum_{j\in N} f_{ij} x_j^* \qquad (28)$$

and

$$\sum_{j\in N} f_{ij} x_j^* < f_i, \qquad (29)$$

which is a contradiction with the Gomory cut.

Gomory cuts – property 2

Consider an arbitrary integral feasible solution and rewrite the constraint as

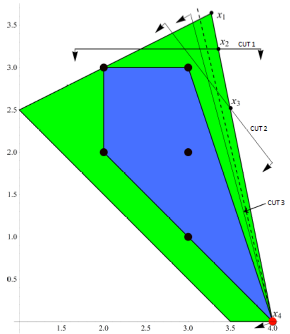
$$x_i + \sum_{j\in N} \lfloor w_{ij} \rfloor x_j - \lfloor d_i \rfloor = f_i - \sum_{j\in N} f_{ij} x_j, \qquad (30)$$

Left-hand side (LS) is integral, thus right-hand side (RS) is integral. Moreover, $f_i < 1$ a $\sum_{j\in N} f_{ij} x_j \ge 0$, thus RS is strictly lower than 1 and at the same time it is integral, thus lower or equal to 0, i.e. we obtain Gomory cut

$$f_i - \sum_{j\in N} f_{ij} x_j \le 0. \qquad (31)$$

Thus each integral solution fulfills it.

Cutting plane methods – steps



Škoda (2010)

Dantzig cuts

$$\sum_{j \in N} x_j \geq 1. \quad (32)$$

(Remind that non-basic variables are equal to zero.)

After two iterations of the dual SIMPLEX algorithm ...

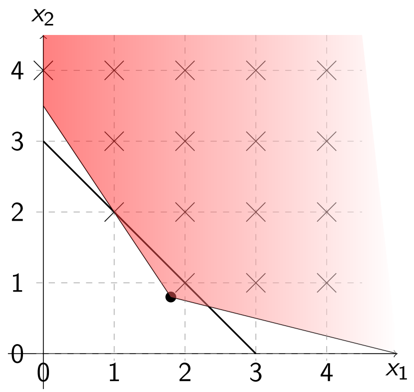
			4	5	0	0
			x_1	x_2	x_3	x_4
5	x_2	8/10	0	1	-3/10	1/10
4	x_1	18/10	1	0	2/10	-4/10
		112/10	0	0	-7/10	-11/10

For example, x_1 is not integral:

$$x_1 + 2/10x_3 - 4/10x_4 = 18/10,$$
$$x_1 + (0 + 2/10)x_3 + (-1 + 6/10)x_4 = 1 + 8/10.$$

Gomory cut:

$$2/10x_3 + 6/10x_4 \geq 8/10.$$



New simplex table

			4	5	0	0	0
			x_1	x_2	x_3	x_4	x_5
5	x_2	8/10	0	1	-3/10	1/10	0
4	x_1	18/10	1	0	2/10	-4/10	0
0	x_5	-8/10	0	0	- 2/10	-6/10	1
		112/10	0	0	-7/10	-11/10	0

Dual simplex alg. ... Gomory cut:

$$4/6x_3 + 1/6x_5 \geq 2/3.$$

Dual simplex alg. ... optimal solution $(2, 1, 1, 1, 0, 0)$.

Literature

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