

# Introduction to Integer Linear Programming

Martin Branda

Charles University  
Faculty of Mathematics and Physics  
Department of Probability and Mathematical Statistics

COMPUTATIONAL ASPECTS OF OPTIMIZATION

# Content

- 1 Motivation and applications
- 2 Formulation and properties
- 3 Cutting plane method

# Knapsack problem

Values  $a_1 = 4$ ,  $a_2 = 6$ ,  $a_3 = 7$ , costs  $c_1 = 4$ ,  $c_2 = 5$ ,  $c_3 = 11$ , budget  $b = 10$ :

$$\begin{aligned} \max \quad & \sum_{i=1}^3 c_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^3 a_i x_i \leq 10, \\ & x_i \in \{0, 1\}. \end{aligned}$$

Consider  $=$  instead of  $\leq$ ,  $0 \leq x_i \leq 1$  and rounding instead of  $x_i \in \{0, 1\}$ , heuristic (ratio  $c_i/a_i$ ) ...

# Why is integrality so important?

Real (mixed-)integer programming problems (not always linear)

- **Portfolio optimization** – integer number of assets, fixed transaction costs
- **Scheduling** – integer (binary) decision variables to assign a job to a machine
- **Vehicle Routing Problems (VRP)** – binary decision variables which identify a successor of a node on the route
- ...

In general – modelling of **logical relations**, e.g.

- at least two constraints from three are fulfilled,
- if we buy this asset then the fixed transaction costs increase,
- ...

# Facility Location Problem

- $i$  warehouses (facilities, branches),  $j$  customers,
- $x_{ij}$  – sent (delivered, served) quantity,
- $y_i$  – a warehouse is built,
- $c_{ij}$  – unit supplying costs,
- $f_i$  – fixed costs of building the warehouse,
- $K_i$  – warehouse capacity,
- $D_j$  – demand.

$$\begin{aligned}
 & \min_{x_{ij}, y_i} \sum_{i=1}^n \sum_{j=1}^m c_{ij} x_{ij} + \sum_i f_i y_i \\
 & \text{s.t.} \quad \sum_{j=1}^m x_{ij} \leq K_i y_i, \quad i = 1, \dots, n, \\
 & \quad \quad \sum_{i=1}^n x_{ij} = D_j, \quad j = 1, \dots, m, \\
 & \quad \quad x_{ij} \geq 0, \quad y_i \in \{0, 1\}.
 \end{aligned}$$

# Scheduling to Minimize the Makespan

- $i$  machines,  $j$  jobs,
- $y$  – machine makespan,
- $x_{ij}$  – assignment variable,
- $t_{ij}$  – time necessary to process job  $j$  on machine  $i$ .

$$\begin{aligned}
 & \min_{x_{ij}, y} y \\
 & \text{s.t.} \quad \sum_{i=1}^m x_{ij} = 1, \quad j = 1, \dots, n, \\
 & \quad \quad \sum_{j=1}^n t_{ij} x_{ij} \leq y, \quad i = 1, \dots, m, \\
 & \quad \quad x_{ij} \in \{0, 1\}, \quad y \geq 0.
 \end{aligned} \tag{1}$$

# Lot Sizing Problem

## Uncapacitated single item LSP

- $x_t$  – production at period  $t$ ,
- $y_t$  – on/off decision at period  $t$ ,
- $s_t$  – inventory at the end of period  $t$  ( $s_0 \geq 0$  fixed),
- $D_t$  – (predicted) *expected* demand at period  $t$ ,
- $p_t$  – unit production costs at period  $t$ ,
- $f_t$  – setup costs at period  $t$ ,
- $h_t$  – inventory costs at period  $t$ ,
- $M$  – a large constant.

$$\begin{aligned}
 \min_{x_t, y_t, s_t} \quad & \sum_{t=1}^T (p_t x_t + f_t y_t + h_t s_t) \\
 \text{s.t.} \quad & s_{t-1} + x_t - D_t = s_t, \quad t = 1, \dots, T, \\
 & x_t \leq M y_t, \\
 & x_t, s_t \geq 0, \quad y_t \in \{0, 1\}.
 \end{aligned} \tag{2}$$

ASS. Wagner-Whitin costs  $p_{t+1} \leq p_t + h_t$ .

# Lot Sizing Problem

## Capacitated single item LSP

- $x_t$  – production at period  $t$ ,
- $y_t$  – on/off decision at period  $t$ ,
- $s_t$  – inventory at the end of period  $t$  ( $s_0 \geq 0$  fixed),
- $D_t$  – (predicted) *expected* demand at period  $t$ .
- $p_t$  – unit production costs at period  $t$ ,
- $f_t$  – setup costs at period  $t$ ,
- $h_t$  – inventory costs at period  $t$ ,
- $C_t$  – production capacity at period  $t$ .

$$\begin{aligned}
 \min_{x_t, y_t, s_t} \quad & \sum_{t=1}^T (p_t x_t + f_t y_t + h_t s_t) \\
 \text{s.t.} \quad & s_{t-1} + x_t - D_t = s_t, \quad t = 1, \dots, T, \\
 & x_t \leq C_t y_t, \\
 & x_t, s_t \geq 0, \quad y_t \in \{0, 1\}.
 \end{aligned} \tag{3}$$

ASS. Wagner-Whitin costs  $p_{t+1} \leq p_t + h_t$ .



# Unit Commitment Problem

- $i = 1, \dots, n$  units (power plants),  $t = 1, \dots, T$  periods,
- $y_{it}$  – on/off decision for unit  $i$  at period  $t$ ,
- $x_{it}$  – production level for unit  $i$  at period  $t$ ,
- $D_t$  – (predicted) *expected* demand at period  $t$ ,
- $p_i^{\min}, p_i^{\max}$  – minimal/maximal production capacity of unit  $i$ ,
- $c_{it}$  – variable production costs,
- $f_{it}$  – (fixed) start-up costs.

$$\begin{aligned}
 & \min_{x_{it}, y_{it}} \sum_{i=1}^n \sum_{t=1}^T (c_{it} x_{it} + f_{it} y_{it}) \\
 & \text{s.t.} \quad \sum_{i=1}^n x_{it} \geq D_t, \quad t = 1, \dots, T, \\
 & \quad p_i^{\min} y_{it} \leq x_{it} \leq p_i^{\max} y_{it}, \\
 & \quad x_{it} \geq 0, \quad y_{it} \in \{0, 1\}.
 \end{aligned} \tag{4}$$

# Sparse $l_1$ regression

- $Y_i$  – dependent variable  $i = 1, \dots, n$ ,
- $X_{ij}$  – explanatory (independent) variables  $j = 1, \dots, m$ ,
- $\beta_j$  – coefficients.

$$\min_{\beta_j} \sum_{i=1}^n \left| Y_i - \sum_{j=1}^m X_{ij} \beta_j \right| \quad (5)$$

s.t. at most  $\kappa < m$  coefficients are nonzero.

MILP reformulation

$$\begin{aligned} \min_{\beta, u, z} \quad & \sum_{i=1}^n u_i^+ + u_i^- \\ \text{s.t.} \quad & u_i^+ - u_i^- = Y_i - \sum_{j=1}^m X_{ij} \beta_j, \\ & -Mz_j \leq \beta_j \leq Mz_j, \\ & \sum_{j=1}^m z_j \leq \kappa, \quad u_i^+, u_i^- \geq 0, \quad z_j \in \{0, 1\}. \end{aligned} \quad (6)$$

# Chance constrained problems – single random constraint

Let  $f, g(\cdot, \xi) : \mathbb{R}^n \rightarrow \mathbb{R}$  be real functions,  $X \subseteq \mathbb{R}^n$ ,  $\xi$  be a real random vector,  $\varepsilon \in (0, 1)$  small:

$$\begin{array}{ll} \min_{x \in X} & f(x) \\ \text{s.t.} & P(g(x, \xi) \leq 0) \geq 1 - \varepsilon. \end{array}$$

INTERPRETATION: for a given  $x \in X$ , the probability of  $\xi$  for which the random constraint is fulfilled must be at least  $1 - \varepsilon$ :

$$P(g(x, \xi) \leq 0) = P(\{\xi : g(x, \xi) \leq 0\}).$$

# Chance constrained problems – single random constraint

Let  $\xi$  have a finite discrete distribution with realizations  $\xi^1, \dots, \xi^S$  and probabilities  $p_s > 0$ ,  $\sum_{s=1}^S p_s = 1$ :

$$\begin{aligned}
 & \min_{x,y} f(x) \\
 & \text{s.t.} \\
 & \sum_{s=1}^S p_s y_s \geq 1 - \varepsilon, \\
 & g(x, \xi_s) \leq M(1 - y_s), \quad s = 1, \dots, S \\
 & y_s \in \{0, 1\}, \quad s = 1, \dots, S, \\
 & x \in X,
 \end{aligned} \tag{7}$$

where  $M \geq \max_{s=1, \dots, S} \sup_{x \in X} g(x, \xi_s)$ .

Example: Value at Risk (VaR).

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# Integer linear programming

$$\min c^T x \quad (8)$$

$$Ax \geq b, \quad (9)$$

$$x \in \mathbb{Z}_+^n. \quad (10)$$

**Assumption:** all coefficients are integer (rational before multiplying by a proper constant).

## Set of feasible solution and its relaxation

$$S = \{x \in \mathbb{Z}_+^n : Ax \geq b\}, \quad (11)$$

$$P = \{x \in \mathbb{R}_+^n : Ax \geq b\} \quad (12)$$

Obviously  $S \subseteq P$ . Not so trivial that  $S \subseteq \text{conv}(S) \subseteq P$ .

## ILP – irrational data

Škoda (2010):

$$\begin{aligned} \max \quad & \sqrt{2}x - y \\ \text{s.t.} \quad & \sqrt{2}x - y \leq 0, \\ & x \geq 1, \\ & x, y \in \mathbb{N}. \end{aligned} \tag{13}$$

The objective value is bounded (from above), but there is no optimal solution.

For any feasible solution with the objective value  $z = \sqrt{2}x^* - \lceil \sqrt{2}x^* \rceil$  we can construct a solution with a higher objective value...

# ILP – irrational data

Let  $z = \sqrt{2}x^* - \lceil \sqrt{2}x^* \rceil$  be the optimal solution. Since  $-1 < z < 0$ , we can find  $k \in \mathbb{N}$  such that  $kz < -1$  and  $(k-1)z > -1$ . By setting  $\epsilon = -1 - kz$  we get that  $-1 < z < -\epsilon = 1 + kz < 0$ . Then

$$\begin{aligned}
 & \sqrt{2}kx^* - \lceil \sqrt{2}kx^* \rceil \\
 &= kz + k \lceil \sqrt{2}x^* \rceil - \lceil \sqrt{2}kx^* \rceil \\
 &= -1 - \epsilon + k \lceil \sqrt{2}x^* \rceil - \lceil \sqrt{2}kx^* \rceil \\
 &= k \lceil \sqrt{2}x^* \rceil - 1 - \epsilon - \lceil k \lceil \sqrt{2}x^* \rceil - 1 - \epsilon \rceil \\
 &= -\epsilon > z.
 \end{aligned} \tag{14}$$

( $k \lceil \sqrt{2}x^* \rceil - 1$  is integral)

Thus, we have obtained a solution with a higher objective value which is a contradiction.

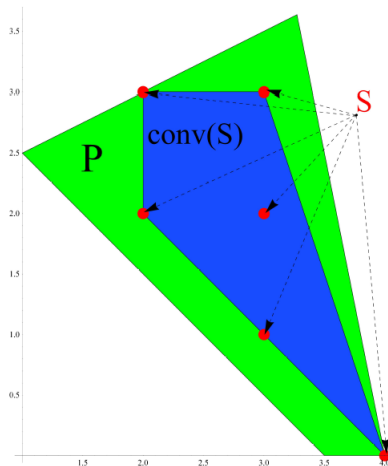


# Example

Consider set  $S$  given by

$$\begin{aligned} 7x_1 + 2x_2 &\geq 5, \\ 7x_1 + x_2 &\leq 28, \\ -4x_1 + 14x_2 &\leq 35, \\ x_1, x_2 &\in \mathbb{Z}_+. \end{aligned}$$

# Set of feasible solutions, its relaxation and convex envelope



Škoda (2010)

# Integer linear programming problem

Problem

$$\min c^T x : x \in S. \quad (15)$$

is equivalent to

$$\min c^T x : x \in \text{conv}(S). \quad (16)$$

$\text{conv}(S)$  is very difficult to construct – many constraints ("strong cuts") are necessary (there are some important exceptions).

LP-relaxation:

$$\min c^T x : x \in P. \quad (17)$$

# Mixed-integer linear programming

Often both integer and continuous decision variables appear:

$$\begin{aligned} \min \quad & c^T x + d^T y \\ \text{s.t.} \quad & Ax + By \geq b \\ & x \in \mathbb{Z}_+^n, y \in \mathbb{R}_+^{n'}. \end{aligned}$$

(WE DO NOT CONSIDER IN INTRODUCTION)

# Basic algorithms

We consider:

- **Cutting Plane Method**
- **Branch-and-Bound**

There are methods which combine the previous alg., e.g.

**Branch-and-Cut** (add cuts to reduce the problem for B&B).

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# Cutting plane method – Gomory cuts

1. Solve LP-relaxation using (primal or dual) SIMPLEX algorithm.
  - If the solution is integral – END, we have found an optimal solution,
  - otherwise continue with the next step.
2. Add a **Gomory cut** (...) and solve the resulting problem using DUAL SIMPLEX alg.

# Example

$$\min 4x_1 + 5x_2 \tag{18}$$

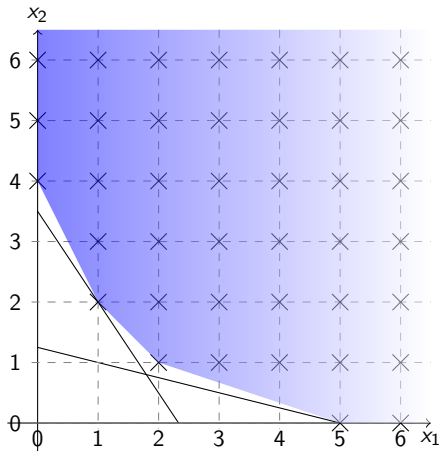
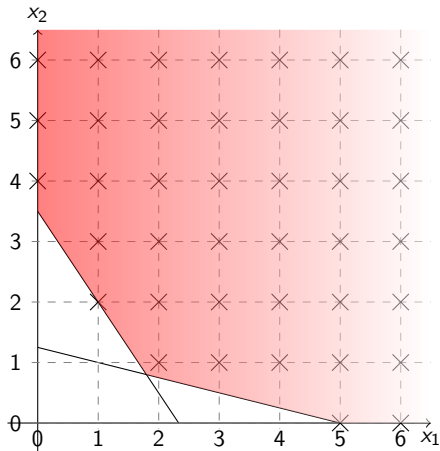
$$x_1 + 4x_2 \geq 5, \tag{19}$$

$$3x_1 + 2x_2 \geq 7, \tag{20}$$

$$x_1, x_2 \in \mathbb{Z}_+^n. \tag{21}$$

Dual simplex for LP-relaxation ...





After two iterations of the dual SIMPLEX algorithm ...

			4	5	0	0
			$x_1$	$x_2$	$x_3$	$x_4$
5	$x_2$	8/10	0	1	-3/10	1/10
4	$x_1$	18/10	1	0	2/10	-4/10
		112/10	0	0	-7/10	-11/10

# Gomory cuts

There is a row in simplex table, which corresponds to a **non-integral solution**  $x_i$  in the form:

$$x_i + \sum_{j \in N} w_{ij} x_j = d_i, \quad (22)$$

where  $N$  denotes the set of non-basic variables;  $d_i$  is non-integral. We denote

$$w_{ij} = \lfloor w_{ij} \rfloor + f_{ij}, \quad (23)$$

$$d_i = \lfloor d_i \rfloor + f_i, \quad (24)$$

i.e.  $0 \leq f_{ij}, f_i < 1$ .

$$\sum_{j \in N} f_{ij} x_j \geq f_i, \quad (25)$$

or rather  $-\sum_{j \in N} f_{ij} x_j + s = -f_i, s \geq 0$ .

# Gomory cuts

General properties of cuts (including Gomory ones):

- Property 1: Current (non-integral) solution becomes infeasible (it is cut).
- Property 2: No feasible integral solution becomes infeasible (it is not cut).

# Gomory cuts – property 1

We express the constraints in the form

$$x_i + \sum_{j \in N} (\lfloor w_{ij} \rfloor + f_{ij}) x_j = \lfloor d_i \rfloor + f_i, \quad (26)$$

$$x_i + \sum_{j \in N} \lfloor w_{ij} \rfloor x_j - \lfloor d_i \rfloor = f_i - \sum_{j \in N} f_{ij} x_j. \quad (27)$$

Current solution  $x_j^* = 0$  for  $j \in N$  and  $x_i^* = d_i$  is non-integral, i.e.  $0 < x_i^* - \lfloor d_i \rfloor < 1$ , thus

$$0 < x_i^* - \lfloor d_i \rfloor = f_i - \sum_{j \in N} f_{ij} x_j^* \quad (28)$$

and

$$\sum_{j \in N} f_{ij} x_j^* < f_i, \quad (29)$$

which is a contradiction with the Gomory cut.

# Gomory cuts – property 2

Consider an arbitrary integral feasible solution and rewrite the constraint as

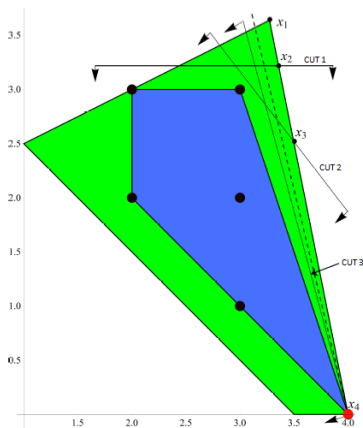
$$x_i + \sum_{j \in N} \lfloor w_{ij} \rfloor x_j - \lfloor d_i \rfloor = f_i - \sum_{j \in N} f_{ij} x_j, \quad (30)$$

Left-hand side (LS) is integral, thus right-hand side (RS) is integral. Moreover,  $f_i < 1$  a  $\sum_{j \in N} f_{ij} x_j \geq 0$ , thus RS is strictly lower than 1 and at the same time it is integral, thus lower or equal to 0, i.e. we obtain Gomory cut

$$f_i - \sum_{j \in N} f_{ij} x_j \leq 0. \quad (31)$$

Thus each integral solution fulfills it.

# Cutting plane methods – steps



Škoda (2010)

# Dantzig cuts

$$\sum_{j \in N} x_j \geq 1. \quad (32)$$

(Remind that non-basic variables are equal to zero.)



After two iterations of the dual SIMPLEX algorithm ...

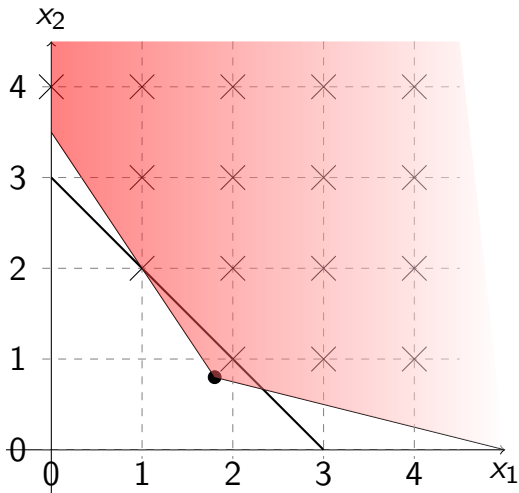
			4	5	0	0
			$x_1$	$x_2$	$x_3$	$x_4$
5	$x_2$	8/10	0	1	-3/10	1/10
4	$x_1$	18/10	1	0	2/10	-4/10
		112/10	0	0	-7/10	-11/10

For example,  $x_1$  is not integral:

$$\begin{aligned} x_1 + 2/10x_3 - 4/10x_4 &= 18/10, \\ x_1 + (0 + 2/10)x_3 + (-1 + 6/10)x_4 &= 1 + 8/10. \end{aligned}$$

Gomory cut:

$$2/10x_3 + 6/10x_4 \geq 8/10.$$



New simplex table

			4	5	0	0	0
			$x_1$	$x_2$	$x_3$	$x_4$	$x_5$
5	$x_2$	8/10	0	1	-3/10	1/10	0
4	$x_1$	18/10	1	0	2/10	-4/10	0
0	$x_5$	-8/10	0	0	-2/10	-6/10	1
		112/10	0	0	-7/10	-11/10	0

Dual simplex alg. ... Gomory cut:

$$4/6x_3 + 1/6x_5 \geq 2/3.$$

Dual simplex alg. ... optimal solution (2, 1, 1, 1, 0, 0).

# Literature

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