

Předmět: NMTM102 Matematická analýza II

Typ výuky: Cvičení

Vahid Borji

borji@karlin.mff.cuni.cz

Taylorův polynom n -tého řádu funkce f v bodě $a \in \mathbb{R}$:

$$T_n(x) = f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f^{(3)}(a)}{3!} (x - a)^3 + \dots + \frac{f^{(n)}(a)}{n!} (x - a)^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3), x \rightarrow 0$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5), x \rightarrow 0$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4), x \rightarrow 0$$

$$(1 + x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1} \cdot x + \binom{\alpha}{2} x^2 + \dots + \binom{\alpha}{n} x^n + o(x^n), x \rightarrow 0$$

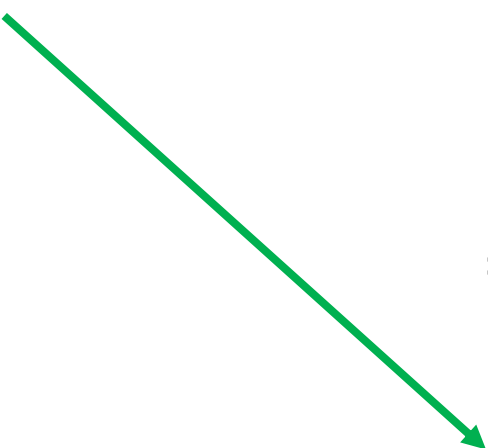
$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + o(x^5), x \rightarrow 0$$

$$\arctg x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + o(x^7), x \rightarrow 0$$

$$(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1} \cdot x + \binom{\alpha}{2} x^2 + \dots + \binom{\alpha}{n} x^n + o(x^n), x \rightarrow 0$$

$$\sqrt[3]{1+x} = (1+x)^{\frac{1}{3}} = \binom{\frac{1}{3}}{0} + \binom{\frac{1}{3}}{1} x + \binom{\frac{1}{3}}{2} x^2 + o(x^2), x \rightarrow 0$$

$$= 1 + \frac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!} x^2 + o(x^2), x \rightarrow 0$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + o(x^2), x \rightarrow 0$$


$$\lim_{x \rightarrow 0} \frac{2\sin(\sin x) - \sin(2x)\sqrt[3]{1+x^2}}{x^5}$$

$$\sin y = y - \frac{y^3}{6} + \frac{y^5}{120} + o(y^5), y \rightarrow 0$$

$$\boxed{y = \sin x} \quad \sin(\sin x) = \sin x - \frac{(\sin x)^3}{6} + \frac{(\sin x)^5}{120} + o(x^5), x \rightarrow 0$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right) - \frac{\left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right)^3}{6} + \frac{\left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right)^5}{120} + o(x^5), x \rightarrow 0$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^3 - 3\frac{x^5}{6} + o(x^5)}{6} + \frac{x^5 + o(x^5)}{120}, x \rightarrow 0$$

$$= x + x^3\left(-\frac{1}{6} - \frac{1}{6}\right) + x^5\left(\frac{1}{120} + \frac{1}{12} + \frac{1}{120}\right) + o(x^5), x \rightarrow 0 = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 + o(x^5), x \rightarrow 0$$

$$\lim_{x \rightarrow 0} \frac{2\sin(\sin x) - \sin(2x)\sqrt[3]{1+x^2}}{x^5} \quad \sin(\sin x) = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 + o(x^5), x \rightarrow 0$$

$$\sin(y) = y - \frac{y^3}{6} + \frac{y^5}{120} + o(y^5), y \rightarrow 0 \quad \sin(2x) = 2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{120} + o(x^5), x \rightarrow 0$$

$$\sqrt[3]{1+y} = (1+y)^{\frac{1}{3}} = \binom{\frac{1}{3}}{0} + \binom{\frac{1}{3}}{1}y + \binom{\frac{1}{3}}{2}y^2 + o(y^2), y \rightarrow 0 = 1 + \frac{1}{3}y - \frac{1}{9}y^2 + o(y^2), y \rightarrow 0$$

$$\sqrt[3]{1+x^2} = 1 + \frac{1}{3}x^2 - \frac{1}{9}(x^2)^2 + o(x^4), x \rightarrow 0 = 1 + \frac{1}{3}x^2 - \frac{1}{9}x^4 + o(x^4), x \rightarrow 0$$

$$\sin(2x)\sqrt[3]{1+x^2} = 2x - \frac{8x^3}{6} + \frac{32x^5}{120} + \frac{2}{3}x^3 - \frac{8x^5}{18} - \frac{2}{9}x^5 + o(x^5), x \rightarrow 0$$

$$= 2x - \frac{2}{3}x^3 - \frac{2x^5}{5} + o(x^5), x \rightarrow 0$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{2x} - \frac{2}{3}x^3 + \frac{1}{5}x^5 - \cancel{2x} + \frac{2}{3}x^3 + \frac{2x^5}{5} + o(x^5)}{x^5} = \lim_{x \rightarrow 0} \frac{\frac{3x^5}{5}}{x^5} + \lim_{x \rightarrow 0} \frac{o(x^5)}{x^5} = \frac{3}{5} + 0 = \frac{3}{5}$$

$$\lim_{x \rightarrow 0} \frac{e^{-2x^2} - \cos 2x}{\sqrt{1+x} - \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}\right)}$$

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{1}x + \binom{\frac{1}{2}}{2}x^2 + \binom{\frac{1}{2}}{3}x^3 + \binom{\frac{1}{2}}{4}x^4 + o(x^4), x \rightarrow 0$$

$$\sqrt{1+x} = 1 + \frac{\frac{1}{2}}{1!}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}x^4 + o(x^4), x \rightarrow 0$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5}{128}x^4 + o(x^4), x \rightarrow 0$$

$$e^y = 1 + y + \frac{y^2}{2} + o(y^2), y \rightarrow 0$$

$$\boxed{y = -2x^2} \quad e^{-2x^2} = 1 - 2x^2 + \frac{(-2x^2)^2}{2} + o(x^4), x \rightarrow 0 = 1 - 2x^2 + 2x^4 + o(x^4), x \rightarrow 0$$

$$\lim_{x \rightarrow 0} \frac{e^{-2x^2} - \cos 2x}{\sqrt{1+x} - \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}\right)}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5}{128}x^4 + o(x^4), x \rightarrow 0$$

$$e^{-2x^2} = 1 - 2x^2 + 2x^4 + o(x^4), x \rightarrow 0$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} + o(y^4), y \rightarrow 0$$

$$y = 2x \quad \cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + o(x^4), x \rightarrow 0 = 1 - 2x^2 + \frac{16}{24}x^4 + o(x^4), x \rightarrow 0$$

$$\lim_{x \rightarrow 0} \frac{\cancel{1 - 2x^2 + 2x^4} - \cancel{1 - 2x^2} - \frac{16}{24}x^4 + o(x^4)}{\cancel{1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}} - \frac{5}{128}x^4 - \cancel{1 - \frac{x}{2} + \frac{x}{8} - \frac{x^3}{16}} + o(x^4)} = \lim_{x \rightarrow 0} \frac{\frac{4}{3}x^4 + o(x^4)}{-\frac{5}{128}x^4 + o(x^4)}$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{x^4} \left(\frac{4}{3} + \frac{o(x^4)}{x^4} \right) \text{ VOAL}}{\cancel{x^4} \left(-\frac{5}{128} + \frac{o(x^4)}{x^4} \right)} = \frac{\frac{4}{3} + \lim_{x \rightarrow 0} \frac{o(x^4)}{x^4} \overset{0}{}}{-\frac{5}{128} + \lim_{x \rightarrow 0} \frac{o(x^4)}{x^4} \overset{0}{}} = \frac{\frac{4}{3}}{-\frac{5}{128}} = -\frac{512}{15}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - \ln(1+x) - 1}{x^3}$$

$$\sqrt{1+y} = 1 + \frac{y}{2} - \frac{y^2}{8} + \frac{y^3}{16} + o(y^3), y \rightarrow 0$$

$$y = 2x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + o(x^5), x \rightarrow 0$$

$$\sqrt{1+2x} = 1 + \frac{2x}{2} - \frac{(2x)^2}{8} + \frac{(2x)^3}{16} + o(x^3), x \rightarrow 0 = 1 + x - \frac{x^2}{2} + \frac{x^3}{2} + o(x^3), x \rightarrow 0$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{1} + \cancel{x} - \cancel{x^2}/2 + \frac{x^3}{2} - \cancel{x} + \cancel{x^2}/2 - \frac{x^3}{3} - \cancel{1} + o(x^3)}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{6} + o(x^3)}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{x^3} \left(\frac{1}{6} + \frac{o(x^3)}{x^3} \right)}{\cancel{x^3}} = \frac{1}{6}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 + \ln\left(1 - \frac{x^2}{2}\right)} - 1 + \frac{x^2}{4}}{x^4}$$

$$\sqrt{1 + y} = 1 + \frac{y}{2} - \frac{y^2}{8} + \frac{y^3}{16} - \frac{5}{128}y^4 + o(y^4), y \rightarrow 0$$

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} + o(z^5), z \rightarrow 0$$

$$z = -\frac{x^2}{2}$$

$$\ln\left(1 - \frac{x^2}{2}\right) = -\frac{x^2}{2} - \frac{\left(-\frac{x^2}{2}\right)^2}{2} + o(x^4), x \rightarrow 0 = -\frac{x^2}{2} - \frac{x^4}{8} + o(x^4), x \rightarrow 0$$

$$y = \ln\left(1 - \frac{x^2}{2}\right) \quad \sqrt{1 + y} = 1 + \frac{-\frac{x^2}{2} - \frac{x^4}{8}}{2} - \frac{\left(-\frac{x^2}{2} - \frac{x^4}{8}\right)^2}{8} + o(x^4), x \rightarrow 0$$

$$= 1 - \frac{x^2}{4} - \frac{x^4}{16} - \frac{x^4}{32} + o(x^4), x \rightarrow 0$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{1} - \frac{x^2}{4} - \frac{x^4}{16} - \frac{x^4}{32} + o(x^4) - \cancel{1} + \frac{x^2}{4}}{x^4} = \lim_{x \rightarrow 0} \frac{-\frac{3x^4}{32} + o(x^4)}{x^4} = -\frac{3}{32}$$

Najděte $a, b \in \mathbb{R}$ tak, aby $\lim_{x \rightarrow 0} \frac{x - (a + b \cos x) \sin x}{x^4} = 0$

Najděte $a, b \in \mathbb{R}$ tak, aby $\lim_{x \rightarrow 0} \frac{x - (a + b \cos x) \sin x}{x^4} = 0$, $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5), x \rightarrow 0$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4), x \rightarrow 0 \Rightarrow b \cos x = b - b \frac{x^2}{2} + b \frac{x^4}{24} + o(x^4), x \rightarrow 0$$

$$a + b \cos x = a + b - b \frac{x^2}{2} + b \frac{x^4}{24} + o(x^4), x \rightarrow 0$$

$$(a + b \cos x) \sin x = (a + b)x - (a + b) \frac{x^3}{6} + (a + b) \frac{x^5}{120} - b \frac{x^3}{2} + b \frac{x^5}{12} + b \frac{x^5}{24} + o(x^5), x \rightarrow 0$$

$$= (a + b)x - \left(\frac{a + 4b}{6}\right)x^3 + \left(\frac{a + 16b}{120}\right)x^5 + o(x^5), x \rightarrow 0$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{x - (a + b)x} + \left(\frac{a + 4b}{6}\right)x^3 - \left(\frac{a + 16b}{120}\right)x^5 + o(x^5)}{x^4}$$

$$1 - a - b = 0$$

$$a = \frac{4}{3}$$

$$\frac{a + 4b}{6} = 0 \Rightarrow a + 4b = 0$$

$$b = -\frac{1}{3}$$

Lagrangeův tvar zbytku Věta 30 (Taylorova věta s Lagrangeovým tvarem zbytku).

Necht' $n \in \mathbb{N}$, $a, x \in \mathbb{R}$, $a \neq x$.

Budiž f funkce, jež má spojitě derivace až do řádu $n + 1$ na $[a, x]$.

Pak existuje $\xi \in (a, x)$ takové, že:

$$f(x) - T_n^{f,a}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1},$$

kde

Chyba

$$T_n^{f,a}(x) = f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f^{(3)}(a)}{3!} (x - a)^3 + \dots + \frac{f^{(n)}(a)}{n!} (x - a)^n$$

Vypočtěte:

$\sin(0, 1)$ s chybou menší než 10^{-4} .

$$f(x) = \sin x \quad a = 0$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6), x \rightarrow 0 \quad \boxed{x = \frac{1}{10}}$$

$$|f(x) - T_n^{f,0}(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1} \right| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \left(\frac{1}{10}\right)^{n+1} \right| < \left| \frac{1}{(n+1)!} \frac{1}{10^{n+1}} \right|$$
$$= \frac{1}{(n+1)!} \frac{1}{10^{n+1}} < 10^{-4}$$

$$10^4 < (n+1)! 10^{n+1}$$

$n = 1 \quad 10^4 < (2)! 10^2$ ❌

$n = 2 \quad 10^4 < (3)! 10^3$ ❌

$n = 3 \quad 10^4 < (4)! 10^4$ ✅

$$T_3^{f,0}(x) = x - \frac{x^3}{3!} \Rightarrow T_3^{f,0}(0, 1) = 0, 1 - \frac{(0, 1)^3}{3!} = \boxed{0, 09983333}$$

$$\sin(0, 1) = 0, 09983342$$

$$|\sin(0, 1) - T_3^{f,0}(0, 1)| = |0, 09983342 - 0, 09983333| = 0, 00000009 < 10^{-4}$$

Vypočtete: $\sqrt[3]{e}$ s chybou menší než 10^{-4} . $\sqrt[3]{e} = e^{\frac{1}{3}}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n), x \rightarrow 0 \quad a = 0 \quad x = \frac{1}{3}$$

$$\xi \in (0, \frac{1}{3}) \Rightarrow \xi < \frac{1}{3} \Rightarrow e^\xi < e^{\frac{1}{3}}$$

$$|f(x) - T_n^{f,0}(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \left(\frac{1}{3}\right)^{n+1} \right| = \left| \frac{e^\xi}{(n+1)!} \left(\frac{1}{3}\right)^{n+1} \right| < \left| \frac{e^{\frac{1}{3}}}{(n+1)!} \left(\frac{1}{3}\right)^{n+1} \right| < \frac{2}{(n+1)!} \frac{1}{3^{n+1}} < 10^{-4}$$

$$2 \cdot 10^4 < (n+1)! 3^{n+1}$$

$$n = 1 \quad 2 \cdot 10^4 < (2)! 3^2 \Rightarrow 20000 < 18 \quad \times$$

$$n = 2 \quad 2 \cdot 10^4 < (3)! 3^3 \Rightarrow 20000 < 162 \quad \times$$

$$n = 3 \quad 2 \cdot 10^4 < (4)! 3^4 \Rightarrow 20000 < 1994 \quad \times$$

$$n = 4 \quad 2 \cdot 10^4 < (5)! 3^5 \Rightarrow 20000 < 29160 \quad \checkmark$$

$$T_4^{f,0}\left(\frac{1}{3}\right) = 1 + \frac{1}{3} + \frac{\left(\frac{1}{3}\right)^2}{2!} + \frac{\left(\frac{1}{3}\right)^3}{3!} + \frac{\left(\frac{1}{3}\right)^4}{4!}$$

$$\left| e^{\frac{1}{3}} - T_4^{f,0}\left(\frac{1}{3}\right) \right| = |1.3956124 - 1.39557613| < 10^{-4}$$

$$\lim_{x \rightarrow \infty} (\sqrt[6]{x^6 + x^5} - \sqrt[6]{x^6 - x^5}) \quad \boxed{x = \frac{1}{y}} \Rightarrow y = \frac{1}{x} \quad x \rightarrow \infty \Rightarrow y \rightarrow 0^+$$

$$= \lim_{x \rightarrow \infty} \left(x \sqrt[6]{1 + \frac{1}{x}} - x \sqrt[6]{1 - \frac{1}{x}} \right) = \lim_{y \rightarrow 0^+} \left(\frac{1}{y} \sqrt[6]{1 + y} - \frac{1}{y} \sqrt[6]{1 - y} \right)$$

$$\sqrt[6]{1 + y} = (1 + y)^{\frac{1}{6}} = \binom{\frac{1}{6}}{0} + \binom{\frac{1}{6}}{1} y + o(y), y \rightarrow 0 = 1 + \frac{1}{6} y + o(y), y \rightarrow 0$$

$$\sqrt[6]{1 - y} = (1 - y)^{\frac{1}{6}} = \binom{\frac{1}{6}}{0} - \binom{\frac{1}{6}}{1} y + o(y), y \rightarrow 0 = 1 - \frac{1}{6} y + o(y), y \rightarrow 0$$

$$= \lim_{y \rightarrow 0^+} \left(\frac{1}{y} \left(1 + \frac{1}{6} y + o(y) \right) - \frac{1}{y} \left(1 - \frac{1}{6} y + o(y) \right) \right) = \lim_{y \rightarrow 0^+} \left(\cancel{\frac{1}{y}} - \cancel{\frac{1}{y}} + \frac{1}{6} + \frac{1}{6} + o(1) \right) = \frac{1}{3}$$

$$\lim_{x \rightarrow 0} \frac{e^{\cos x - 1} - \cos x}{x(\sin x - x)} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4), x \rightarrow 0 \Rightarrow \cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4), x \rightarrow 0$$

$$y = \cos x - 1$$

$$e^y = 1 + y + \frac{y^2}{2} + o(y^2), y \rightarrow 0 \Rightarrow e^{\cos x - 1} = 1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)\right) + \frac{\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)\right)^2}{2}$$

$$e^{\cos x - 1} = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^4}{8} + o(x^4), x \rightarrow 0 = 1 - \frac{x^2}{2} + \frac{x^4}{6} + o(x^4), x \rightarrow 0 + o(x^4), x \rightarrow 0$$

$$\sin x = x - \frac{x^3}{3!} + o(x^3), x \rightarrow 0 \Rightarrow \sin x - x = -\frac{x^3}{3!} + o(x^3), x \rightarrow 0$$

$$\lim_{x \rightarrow 0} \frac{\cancel{1} - \frac{x^2}{2} + \frac{x^4}{6} - \cancel{1} + \frac{x^2}{2!} - \frac{x^4}{4!} + o(x^4)}{-\frac{x^4}{3!} + o(x^4)}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{x^4}{8} + o(x^4)}{-\frac{x^4}{6} + o(x^4)} = \lim_{x \rightarrow 0} \frac{\cancel{x^4} \left(\frac{1}{8} + \frac{o(x^4)}{x^4}\right) \text{ VOAL}}{\cancel{x^4} \left(-\frac{1}{6} + \frac{o(x^4)}{x^4}\right)}$$

$$= \frac{\frac{1}{8} + \lim_{x \rightarrow 0} \frac{o(x^4)}{x^4} \text{ 0}}{-\frac{1}{6} + \lim_{x \rightarrow 0} \frac{o(x^4)}{x^4} \text{ 0}} = -\frac{6}{8} = -\frac{3}{4}$$

$$\lim_{x \rightarrow 0} \frac{e^{\sin x} - 1 - \sin(e^x - 1)}{x^4}$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + o(x^4), x \rightarrow 0$$

$$\sin(e^x - 1) = x + \frac{x^2}{2} - \frac{5}{24}x^4 + o(x^4), x \rightarrow 0$$

$$\lim_{x \rightarrow 0} \frac{e^{\sin x} - 1 - \sin(e^x - 1)}{x^4} = \lim_{x \rightarrow 0} \frac{\cancel{1} + x + \frac{x^2}{2} - \frac{x^4}{8} - \cancel{1} - x - \frac{x^2}{2} + \frac{5}{24}x^4 + o(x^4)}{x^4}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{12}x^4 + o(x^4)}{x^4} = \lim_{x \rightarrow 0} \frac{\cancel{x^4} \left(\frac{1}{12} + \frac{o(x^4)}{x^4} \right)}{\cancel{x^4}}$$

$$= \frac{1}{12} + \lim_{x \rightarrow 0} \frac{o(x^4)}{x^4} = \frac{1}{12} + 0 = \frac{1}{12}$$