Předmět: NMTM102 Matematická analýza II

Typ výuky: Cvičení

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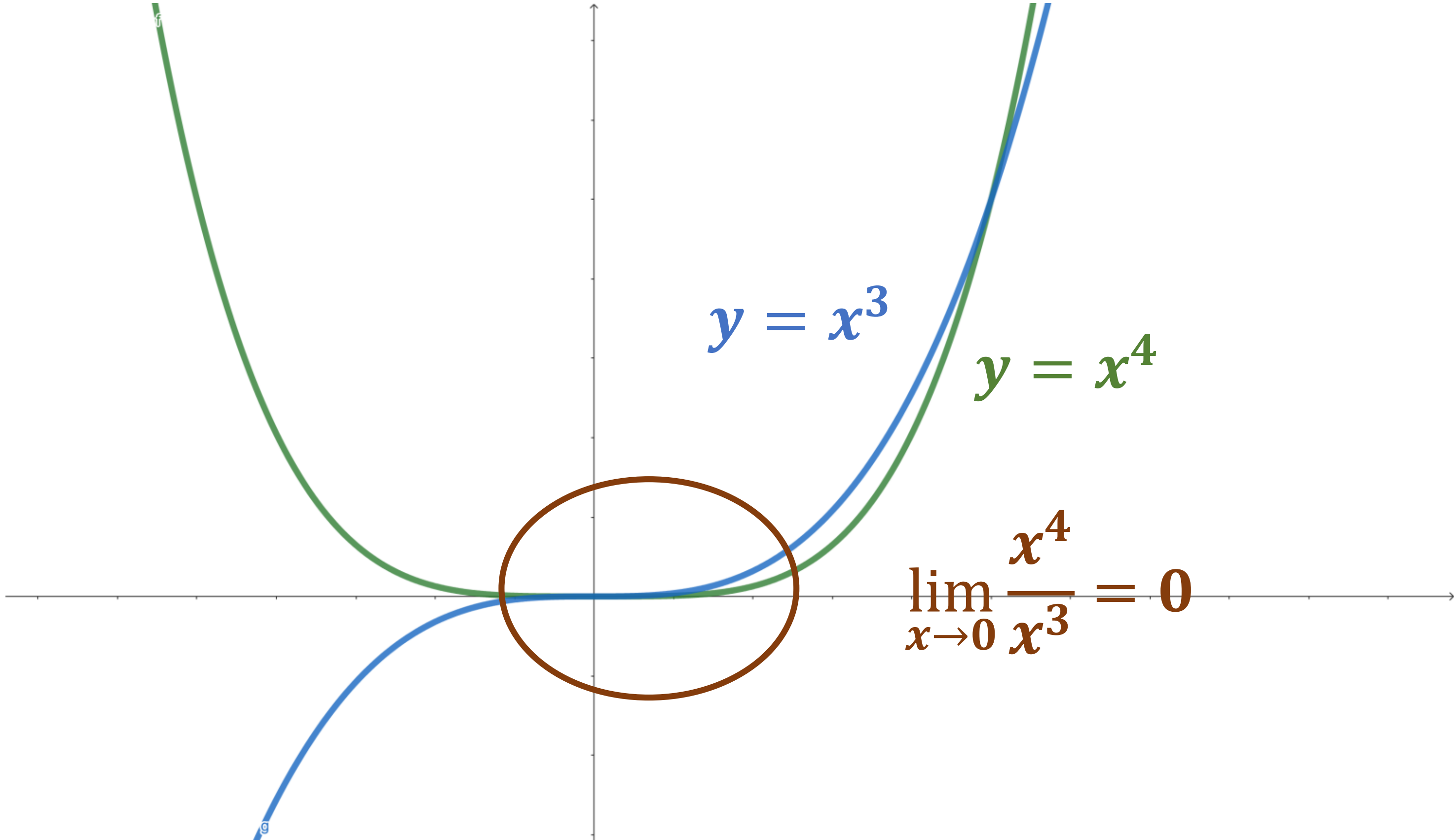
- Symbol malé ***o***
- Taylorův polynom
- Limita funkce

Symbol malé o: Nechť f a g jsou funkce, $a \in \mathbb{R}^*$:

$$f(x) = o(g(x)),\, x \to a \iff \lim_{x \to a} \frac{f(x)}{g(x)} = 0$$

$$x^4 = o\left(x^3\right), x \to 0 \quad \iff \quad \lim_{x \to 0} \frac{x^4}{x^3} = \lim_{x \to a} \frac{x}{1} = 0$$

$$x^4 = o\left(x^3\right), x \to 0$$



✗ $1 = o(x^2), x \to 0$ $\qquad \lim_{x\to 0} \frac{1}{x^2} = \infty$

✗ $x = o(x^2), x \to 0$ $\qquad \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x}$ **neexistuje**

✓ $x^3 = o(x^2), x \to 0 \iff \lim_{x\to 0} \frac{x^3}{x^2} = \lim_{x\to 0} \frac{x}{1} = 0$

✓ $x^4 = o(x^2), x \to 0 \iff \lim_{x\to 0} \frac{x^4}{x^2} = \lim_{x\to 0} \frac{x^2}{1} = 0$

$$\lim_{x\to 0} \frac{x^6}{x^2} = \lim_{x\to 0} x^4 = 0 \iff x^6 = o(x^2), x \to 0$$

✓ $x^5 + x^4 = o(x^3), x \to 0 \iff \lim_{x\to 0} \frac{x^5 + x^4}{x^3} = \lim_{x\to 0} x^2 + x = 0$

Taylorův polynom

Nechť existuje $f^{(n)}(a) \in \mathbb{R}$ **(tedy n-tá derivace je vlastní).**

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

❖ **Definice:** T_n **nazýváme** **Taylorův polynom** **funkce** f **v bodě** a **a řádu** n**.**

Spec.: $a = 0$

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$f(x) \approx T_n(x)$$

$$f(x) = T_n(x) + o(x^n)$$

Nalezněte Taylorův polynom řádu n v bodě 0 pro:

$f(x) = sinx$, $a = 0$, $n = 8$

i	$f^{(i)}(x)$	$f^{(i)}(0)$	$\frac{f^{(i)}(0)}{i!}$
0	$sinx$	0	0
1	$cosx$	1	1
2	$-sinx$	0	0
3	$-cosx$	-1	$-\frac{1}{3!}$
4	$sinx$	0	0
5	$cosx$	1	$\frac{1}{5!}$
6	$-sinx$	0	0
7	$-cosx$	-1	$-\frac{1}{7!}$
8	$sinx$	0	0

$$T_8 = 0 + 1.x + 0.x^2 - \frac{1}{3!}x^3 + 0.x^4 + \frac{1}{5!}x^5 + 0.x^6 - \frac{1}{7!}x^7 + 0.x^8$$

$$T_8 = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$

Taylorova věta

$$sinx = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + o(x^7), x \to 0$$

$$+o(x^8), x \to 0$$

Spočtěte limity:

$$\lim_{x\to 0}\frac{\sin x}{x}$$

$$sinx = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + o\left(x^8\right), x \to 0$$

$$sinx = x + o(x)\ , x \to 0$$

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x + o(x)}{x} = \lim_{x\to 0}\frac{x(1 + \frac{o(x)}{x})}{x} = \lim_{x\to 0}(1 + \frac{o(x)}{x})$$

$$= 1 + \lim_{x\to 0}\frac{o(x)}{x}$$

$$= 1 + 0 = 1$$

$$\lim_{x\to 0}\frac{\sin x - x}{x^3}$$

$$sinx = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + o(x^8), x \to 0$$

$$sinx = x - \frac{x^3}{6} + o(x^3)\ , x \to 0$$

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{\left(x - \frac{x^3}{6} + o(x^3)\right) - x}{x^3} = \lim_{x\to 0}\frac{-\frac{x^3}{6} + o(x^3)}{x^3}$$

$$= \lim_{x\to 0}\frac{-\frac{x^3}{6}}{x^3} + \lim_{x\to 0}\frac{o(x^3)}{x^3} = -\frac{1}{6} + \ 0 = -\frac{1}{6}$$

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} \overset{L.H}{=} \lim_{x\to 0}\frac{cosx - 1}{3x^2} \overset{L.H}{=} \lim_{x\to 0}\frac{-sinx}{6x} \overset{L.H}{=} \lim_{x\to 0}\frac{-cosx}{6} = \frac{-cos0}{6} = -\frac{1}{6}$$

$$\lim_{x\to 0}\frac{\sin x-\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)}{x^7}$$

$$\sin x = x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+o\left(x^8\right), x\to 0$$

$$=\lim_{x\to 0}\frac{x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+o\left(x^8\right)-\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)}{x^7}$$

$$=\lim_{x\to 0}\frac{-\frac{x^7}{5040}+o\left(x^8\right)}{x^7}=\lim_{x\to 0}\frac{-\frac{x^7}{5040}}{x^7}+\lim_{x\to 0}\frac{o\left(x^8\right)}{x^7}=-\frac{1}{5040}$$

$$0$$

$$\lim_{x\to 0}\frac{e^x \sin x - x(1+x)}{x^3}$$

$$e^x = 1 + 1.x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + o(x^4),\ x\to 0$$

i	$f^{(i)}(x)$	$f^{(i)}(0)$	$\frac{f^{(i)}(0)}{i!}$
0	e^x	1	1
1	e^x	1	1
2	e^x	1	$\frac{1}{2!}$
3	e^x	1	$\frac{1}{3!}$
4	e^x	1	$\frac{1}{4!}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4),\ x\to 0$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3),\ x\to 0$$

$$\lim_{x\to 0}\frac{e^x \sin x - x(1+x)}{x^3}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o\left(x^3\right), x \to 0 \qquad sinx = x - \frac{x^3}{6} + o\left(x^3\right), x \to 0$$

$$\lim_{x\to 0}\frac{e^x \sin x - x(1+x)}{x^3} = \lim_{x\to 0}\frac{\overbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o\left(x^3\right)\right)\left(x - \frac{x^3}{6} + o\left(x^3\right)\right)} - x - x^2}{x^3}$$

$$= \lim_{x\to 0}\frac{\cancel{x} - \frac{x^3}{6} + \cancel{x^2} + \frac{x^3}{2!} + o\left(x^3\right) - \cancel{x} - \cancel{x^2}}{x^3} = \lim_{x\to 0}\frac{\frac{x^3}{3} + o\left(x^3\right)}{x^3} = \lim_{x\to 0}\frac{\frac{x^3}{3}}{x^3} + \lim_{x\to 0}\frac{o\left(x^3\right)}{x^3}$$

$$= \frac{1}{3} + 0 = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4}$$

Nalezněte Taylorův polynom řádu n v bodě 0 pro:

$f(x) = cosx \quad , a = 0 \ , n = 9$

i	$f^{(i)}(x)$	$f^{(i)}(0)$	$\frac{f^{(i)}(0)}{i!}$
0	$cosx$	1	1
1	$-sinx$	0	0
2	$-cosx$	-1	$-\frac{1}{2!}$
3	$sinx$	0	0
4	$cosx$	1	$\frac{1}{4!}$
5	$-sinx$	0	0
6	$-cosx$	-1	$-\frac{1}{6!}$
7	$sinx$	0	0
8	$cosx$	1	$\frac{1}{8!}$
9	$-sinx$	0	0

$$T_9 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + 0.x^9$$

Taylorova věta

$$cosx =$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + o(x^8), x \to 0$$

$$+o(x^{10})$$

$$\lim_{x\to 0}\frac{\cos x-e^{-\frac{x^2}{2}}}{x^4}$$

$$\cos x=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+o(x^4),x\to 0$$

$$e^y=1+y+\frac{y^2}{2}+o(y^2),y\to 0$$

$$e^{-\frac{x^2}{2}}=1-\frac{x^2}{2}+\frac{(-\frac{x^2}{2})^2}{2}+o(x^4),x\to 0$$

$$=\lim_{x\to 0}\frac{1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+o(x^4)-(1-\frac{x^2}{2}+\frac{(-\frac{x^2}{2})^2}{2}+o(x^4))}{x^4}$$

$$=\lim_{x\to 0}\frac{\frac{1}{24}x^4-\frac{x^4}{8}+o(x^4)}{x^4}=\lim_{x\to 0}\frac{-\frac{1}{12}x^4+o(x^4)}{x^4}=\lim_{x\to 0}\frac{-\frac{1}{12}x^4}{x^4}+\lim_{x\to 0}\frac{o(x^4)}{x^4}$$

$$=-\frac{1}{12}+0=-\frac{1}{12}$$

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{sinx}\right)=\lim_{x\to 0}\frac{sinx-x}{x.sinx}=\lim_{x\to 0}\frac{\cancel{x}-\frac{x^3}{6}+\frac{x^5}{120}+o(x^5)-\cancel{x}}{x\left(x-\frac{x^3}{6}+o(x^3)\right)}$$

$$=\lim_{x\to 0}\frac{-\frac{x^3}{6}+o(x^3)}{x^2\cancel{-\frac{x^4}{6}+o(x^4)}\ +o(x^2)}=\lim_{x\to 0}\frac{\overset{x}{\cancel{x^3}}\left(-\frac{1}{6}+\frac{o(x^3)}{x^3}\right)}{\cancel{x^2}\left(1+\frac{o(x^2)}{x^2}\right)}$$

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$$=\lim_{x\to 0}x\,\frac{-\frac{1}{6}+\lim_{x\to 0}\frac{o(x^3)}{x^3}}{1+\lim_{x\to 0}\frac{o(x^2)}{x^2}}$$

$$=0\,\frac{-\frac{1}{6}+0}{1+0}=0$$

$$\lim_{x\to 0}\frac{\sin(x^2)-\sin^2 x}{x^4} \qquad \sin y = y-\frac{y^3}{6}+o(y^4),\, y\to 0$$

$$\boxed{y=x^2}\ \sin(x^2)=x^2-\frac{(x^2)^3}{3!}+o(x^8),\, x\to 0 = x^2-\frac{x^6}{3!}+o(x^8),\, x\to 0 = x^2+o(x^4),\, x\to 0$$

$$\sin^2 x = \sin x\,.\,\sin x = \left(x-\frac{x^3}{6}+o(x^4)\right).\left(x-\frac{x^3}{6}+o(x^4)\right) = x^2-\frac{x^4}{6}-\frac{x^4}{6}+o(x^4),\, x\to 0$$

$$= x^2-\frac{x^4}{3}+o(x^4),\, x\to 0$$

$$\lim_{x\to 0}\frac{\sin(x^2)-\sin^2 x}{x^4}$$

$$=\lim_{x\to 0}\frac{x^2-\left(x^2-\frac{x^4}{3}\right)+o(x^4)}{x^4}=\lim_{x\to 0}\frac{\frac{x^4}{3}+o(x^4)}{x^4}=\lim_{x\to 0}\frac{\frac{x^4}{3}}{x^4}+\lim_{x\to 0}\frac{o(x^4)}{x^4}$$

z definice

$$=\frac{1}{3}+0=\frac{1}{3}$$

$$\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)=\cdots=\frac{1}{3}$$