Předmět: NMTM101 Matematická analýza I

Typ výuky: Cvičení

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Cvičení 6

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{2}\right)^n + \frac{1}{n^n} - \left(\frac{1}{3}\right)^n} = \frac{1}{2}$$

$$\frac{1}{2}\cdot\sqrt[n]{\frac{1}{2}} = \sqrt[n]{\frac{1}{2}\cdot\left(\frac{1}{2}\right)^n} = \sqrt[n]{\left(\frac{1}{2}\right)^n - \frac{1}{2}\left(\frac{1}{2}\right)^n} \le \sqrt[n]{\left(\frac{1}{2}\right)^n \underbrace{+ \frac{1}{n^n} - \left(\frac{1}{3}\right)^n}_{<0}} \le \sqrt[n]{\left(\frac{1}{2}\right)^n} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{1}{2}\cdot\sqrt[n]{\frac{1}{2}} = \frac{1}{2}\lim_{n\to\infty}\sqrt[n]{\frac{1}{2}} = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

Věta o dvou policajtech

$$\Rightarrow \quad \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{2}\right)^n + \frac{1}{n^n} - \left(\frac{1}{3}\right)^n} = \frac{1}{2}$$

$$\lim_{n\to\infty}\frac{n^2-13.2^n-\ln n+\sin n}{n^{97}+2^n-e^{n+2}}$$

$$=\lim_{n\to\infty}\frac{2^n\left(\frac{n^2}{2^n}-13-\frac{\ln n}{2^n}+\frac{\sin n}{2^n}\right)}{e^n\left(\frac{n^{97}}{e^n}+\frac{2^n}{e^n}-e^2\right)}=\lim_{n\to\infty}\frac{2^n}{e^n}\lim_{n\to\infty}\frac{\frac{n^2}{2^n}-13-\frac{\ln n}{2^n}+\frac{\sin n}{2^n}}{\frac{n^{97}}{e^n}+\frac{2^n}{e^n}-e^2}$$

R.Š R.Š Nulová . Omezená = Nulová

$$=\lim_{n\to\infty}\left(\frac{2}{e}\right)^n\frac{\lim_{n\to\infty}\frac{n^2}{2^n}-\lim_{n\to\infty}13-\lim_{n\to\infty}\frac{\ln n}{2^n}+\lim_{n\to\infty}\frac{\sin n}{2^n}}{\lim_{n\to\infty}\frac{n^{97}}{e^n}+\lim_{n\to\infty}\frac{2^n}{e^n}-\lim_{n\to\infty}e^2}=0\,\frac{0-13-0+0}{0+0-e^2}=0\,\frac{13}{e^2}=0$$

R.Š R.Š

$$\lim_{n\to\infty}\sqrt[n]{2^{-n}+\left(\frac{1}{10}\right)^{n}+\frac{1}{n}}$$

$$\sqrt[n]{\frac{1}{n}} \le \sqrt[n]{\left(\frac{1}{2}\right)^{n}+\left(\frac{1}{10}\right)^{n}+\frac{1}{n}} \le \sqrt[n]{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}} = \sqrt[n]{\frac{3}{n}}$$

$$\left.\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{\frac{1}{n}} &= \lim_{n\to\infty}\frac{\sqrt[n]{1}}{\sqrt[n]{n}} = \frac{\lim_{n\to\infty}\sqrt[n]{1}}{\lim_{n\to\infty}\sqrt[n]{n}} = \frac{1}{1} = 1 \\
\lim_{n\to\infty}\sqrt[n]{\frac{3}{n}} &= \lim_{n\to\infty}\frac{\sqrt[n]{3}}{\sqrt[n]{n}} = \frac{\lim_{n\to\infty}\sqrt[n]{3}}{\lim_{n\to\infty}\sqrt[n]{n}} = \frac{1}{1} = 1
\end{aligned}\right\} \textbf{Věta o dvou policajtech}$$

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{2}\right)^{n}+\left(\frac{1}{10}\right)^{n}+\frac{1}{n}} = 1$$

$$\lim_{n\to\infty}\frac{\sqrt[2n]{2n^n+2^n+6^n}}{\sqrt{n}+3}$$

$$\frac{\sqrt[2n]{2n^n}}{\sqrt{n}+3}\le\frac{\sqrt[2n]{2n^n+2^n+6^n}}{\sqrt{n}+3}\le\frac{\sqrt[2n]{2n^n+2n^n+2n^n}}{\sqrt{n}+3}=\frac{\sqrt[2n]{6n^n}}{\sqrt{n}+3}$$

$$\lim_{n\to\infty}\frac{\sqrt[2n]{2n^n}}{\sqrt{n}+3}=\lim_{n\to\infty}\frac{\sqrt[2n]{2}\ \sqrt[2n]{n^n}}{\sqrt{n}+3}=\lim_{n\to\infty}\frac{\sqrt[n]{\sqrt{2}}\ \sqrt{n}}{\sqrt{n}+3}=\lim_{n\to\infty}\sqrt[n]{\sqrt{2}}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+3}=1.\lim_{n\to\infty}\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}(1+\frac{3}{\sqrt{n}})}$$

$$=\frac{\lim\limits_{n\to\infty}1}{\lim\limits_{n\to\infty}(1+\frac{3}{\sqrt{n}})}=\frac{1}{1+0}=1$$

$$\lim_{n\to\infty}\frac{\sqrt[2n]{6n^n}}{\sqrt{n}+3}=\lim_{n\to\infty}\frac{\sqrt[2n]{6}\ \sqrt[2n]{n^n}}{\sqrt{n}+3}=\lim_{n\to\infty}\frac{\sqrt[n]{\sqrt{6}}\ \sqrt{n}}{\sqrt{n}+3}=\lim_{n\to\infty}\sqrt[n]{\sqrt{6}}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+3}=1.\lim_{n\to\infty}\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}(1+\frac{3}{\sqrt{n}})}$$

$$=\frac{\lim\limits_{n\to\infty}1}{\lim\limits_{n\to\infty}(1+\frac{3}{\sqrt{n}})}=\frac{1}{1+0}=1$$

Věta o dvou policajtech $\Longrightarrow\ \lim_{n\to\infty}\frac{\sqrt[2n]{2n^n+2^n+6^n}}{\sqrt{n}+3}=1$

$$\lim_{n\to\infty}\frac{n^2.3^n}{4^n}=\lim_{n\to\infty}n^2\left(\frac{3}{4}\right)^n=\lim_{n\to\infty}n^2\frac{1}{\left(\frac{4}{3}\right)^n}=\lim_{n\to\infty}\overset{\text{R.Š}}{\frac{n^2}{\left(\frac{4}{3}\right)^n}}=0$$

$$\lim_{n\to\infty}\frac{n^{1000}.\ln n}{2^n}=\lim_{n\to\infty}\frac{n^{1000}.\ln n}{(\sqrt{2})^n(\sqrt{2})^n}=\lim_{n\to\infty}\overset{\text{R.Š}}{\frac{n^{1000}}{(\sqrt{2})^n}}\lim_{n\to\infty}\overset{\text{R.Š}}{\frac{\ln n}{(\sqrt{2})^n}}=0.0=0$$

$$\lim_{n\to\infty}\frac{-10^{4n}+(2n)!+6.n^n}{13.n^n+3.100^{2n}+\ln n+1}$$

$$n! \ll n^n \ll (2n)!$$

$$n! \ll \frac{(2n)!}{n^n}=\frac{\overbrace{1\,.2\,.3\,\dots\,.(n-1).n}^{=n!}\,.(n+1)\dots.(2n-1).2n}{\underbrace{n}_{>1}\quad\dots\quad\underbrace{n}_{>1}\;.\underbrace{n}_{>1}}$$

$$\lim_{n\to\infty}\frac{-10^{4n}+(2n)!+6.n^n}{13.n^n+3.100^{2n}+\ln n+1}=\infty$$

Definice (Eulerovo číslo).

$$\lim \left(1 + \frac{1}{n}\right)^n = e.$$

$$e = 2{,}718281828\ldots$$

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{n} = \lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n\cdot\frac{1}{2}} = \lim_{n\to\infty}\left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{\frac{1}{2}} = \left(\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}\right)^{\frac{1}{2}} = (e)^{\frac{1}{2}}$$

Veta: **Pokud** $\lim_{n\to\infty} a_n = a \in \mathbb{R}^*$, **pak libovolná vybraná posloupnost z** $\{a_n\}_{n=1}^{\infty}$ **má taky limitu rovnou** a.

$\{(1+\frac{1}{2n})^{2n}\}_{n=1}^{\infty}$ **je vybraná posloupnost z posloupnosti** $\{(1+\frac{1}{n})^{n}\}_{n=1}^{\infty}$

$\longrightarrow e$ $\qquad\qquad$ $\longrightarrow e$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n=\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n=\lim_{n\to\infty}\left(\left(\frac{n}{n-1}\right)^{-1}\right)^n=\lim_{n\to\infty}\left(\left(\frac{n}{n-1}\right)^n\right)^{-1}$$

$$=\left(\lim_{n\to\infty}\left(\frac{n}{n-1}\right)^n\right)^{-1}=\left(\lim_{n\to\infty}\left(\frac{n-1+1}{n-1}\right)^n\right)^{-1}=\left(\lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^n\right)^{-1}$$

$$=\left(\lim_{n\to\infty}\underbrace{\left(1+\frac{1}{n-1}\right)^{n-1}}_{\longrightarrow e}\underbrace{\left(1+\frac{1}{n-1}\right)}_{\longrightarrow 1}\right)^{-1}=e^{-1}$$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+2} = \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^{-1}\right)^{n+2} = \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^{n+2}\right)^{-1} = \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{2}\right)^{-1}$$

$$= \left(\underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}}_{e}\ \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2}}_{1}\right)^{-1} = e^{-1}$$

$$\lim_{n\to\infty} \sqrt[n]{\sqrt{3^n + 3.2^n} + \sqrt{3^n + 2^n}}$$

$$\sqrt[n]{\sqrt{3^n + 3.2^n} + \sqrt{3^n + 2^n}} \leq \sqrt[n]{\sqrt{3^n + 3^n} + \sqrt{3^n + 3^n}} = \sqrt[n]{2\sqrt{2.3^n}} = \sqrt[n]{2\sqrt{2}}\sqrt[n]{\sqrt{3^n}}$$

$$= \sqrt[n]{2\sqrt{2}}\,\sqrt{3}$$

$$\sqrt{3} = \sqrt[n]{\sqrt{3^n}} \leq \sqrt[n]{\sqrt{3^n + 3.2^n} + \sqrt{3^n + 2^n}}$$

$$\lim_{n\to\infty} \sqrt{3} = \sqrt{3}$$

$$\lim_{n\to\infty} \sqrt[n]{2\sqrt{2}}\sqrt{3} = \sqrt{3} \lim_{n\to\infty} \sqrt[n]{2\sqrt{2}} = \sqrt{3}$$

Věta o dvou policajtech

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{\sqrt{3^n + 3.2^n} + \sqrt{3^n + 2^n}} = \sqrt{3}$$

$$\lim_{n\to\infty}\left(\left(arctg\frac{n^2+n+1}{n+2+n^{-2}}\right)\left(\sqrt{3^n+3.2^n}-\sqrt{3^n+2^n}\right)^{\frac{1}{n}}\right)$$

$$=\lim_{n\to\infty}\left(arctg\frac{n^2+n+1}{n+2+n^{-2}}\right)\lim_{n\to\infty}\left(\sqrt{3^n+3.2^n}-\sqrt{3^n+2^n}\right)^{\frac{1}{n}}$$

$$=arctg\left(\lim_{n\to\infty}\frac{n^2+n+1}{n+2+n^{-2}}\right)\lim_{n\to\infty}\left(\left(\sqrt{3^n+3.2^n}-\sqrt{3^n+2^n}\right).\frac{\sqrt{3^n+3.2^n}+\sqrt{3^n+2^n}}{\sqrt{3^n+3.2^n}+\sqrt{3^n+2^n}}\right)^{\frac{1}{n}}$$

$$=arctg(\infty)\lim_{n\to\infty}\left(\frac{\overset{2.2^n}{\cancel{3^n+3.2^n-3^n-2^n}}}{\sqrt{3^n+3.2^n}+\sqrt{3^n+2^n}}\right)^{\frac{1}{n}}=\frac{\pi}{2}\lim_{n\to\infty}\sqrt[n]{\frac{2.2^n}{\sqrt{3^n+3.2^n}+\sqrt{3^n+2^n}}}$$

$$=\frac{\pi}{\cancel{2}}\lim_{n\to\infty}\frac{\cancel{2}.\sqrt[n]{2}}{\sqrt[n]{\sqrt{3^n+3.2^n}+\sqrt{3^n+2^n}}}=\pi\frac{\lim_{n\to\infty}\sqrt[n]{2}}{\lim_{n\to\infty}\sqrt[n]{\sqrt{3^n+3.2^n}+\sqrt{3^n+2^n}}}=\frac{\pi}{\sqrt{3}}$$

$$\lim_{n\to\infty} \frac{\ln^2(n) + 3\log_{10} n + \ln\ln n}{\ln(\ln^4(10n)) + \log_{\frac{1}{3}} n^3 + \ln(\log_{10} n^{10})}$$

$$\ln\ln n \ll \ln n$$

$$= \lim_{n\to\infty} \frac{\ln^2(n) + 3\dfrac{\ln n}{\ln 10} + \ln\ln n}{4\ln(\ln(10n)) + 3\dfrac{\ln n}{\ln\frac{1}{3}} + \ln\left(10\dfrac{\ln n}{\ln 10}\right)}$$

$$= \lim_{n\to\infty} \frac{\cancel{\ln n}\left(\ln n \xrightarrow{} \infty + \dfrac{3}{\ln 10} + \dfrac{\ln\ln n}{\ln n} \xrightarrow{} 0\right)}{\cancel{\ln n}\left(\dfrac{4\ln(\ln(10n))}{\ln n} \xrightarrow{} 0 + \dfrac{3}{\cancel{\ln\frac{1}{3}}_{-\ln 3}} + \dfrac{\ln\left(10\frac{\ln n}{\ln 10}\right)}{\ln n} \xrightarrow{} 0\right)} = \frac{\infty}{\dfrac{3}{-\ln 3}} = -\infty$$