Předmět: NMTM101 Matematická analýza I

Typ výuky: Cvičení

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Cvičení 5

$$(a+b)^{50}=\binom{50}{0}a^{50}+\binom{50}{1}a^{49}b^{1}+\binom{50}{2}a^{48}b^{2}+\cdots+\binom{50}{50}b^{50}$$

$$\lim_{n\to\infty}\frac{(n+1)^{50}-(n-1)^{50}}{n^{49}}=\lim_{n\to\infty}\frac{(\cancel{n^{50}}+50n^{49}+Bn^{48}\ldots+1)-(\cancel{n^{50}}-50n^{49}+Xn^{48}\ldots+1)}{n^{49}}$$

$$=\lim_{n\to\infty}\frac{100n^{49}+(B-X)n^{48}+\ \ldots+?\,n}{n^{49}}=\lim_{n\to\infty}\frac{\cancel{n^{49}}\left(100+\frac{(B-X)}{n}\ldots+\frac{?}{n^{48}}\right)}{\cancel{n^{49}}}=100$$

Věta (Růstová škála). Od jisté hodnoty n platí:

$$\log(\log(\boldsymbol{n})) \ll \mathbf{1000}\log \boldsymbol{n} \ll \boldsymbol{n}^{\frac{\mathbf{1}}{\mathbf{1000}}} \ll \boldsymbol{n} \ll \boldsymbol{n}^{\mathbf{1000}} \ll \mathbf{1{,}0001}^{\boldsymbol{n}} \ll \boldsymbol{n}! \ll \boldsymbol{n}^{\boldsymbol{n}} \ll (\mathbf{2}\boldsymbol{n})!$$

Kde zápis $a_n \ll b_n$ znamená „a_n roste výrazně pomaleji než b_n", tedy:

$$\lim_{\boldsymbol{n}\to\infty} \frac{\boldsymbol{a_n}}{\boldsymbol{b_n}} = \mathbf{0} \qquad \text{a} \quad \lim_{\boldsymbol{n}\to\infty} \frac{\boldsymbol{b_n}}{\boldsymbol{a_n}} = \infty$$

Poučka. Stejně jako při práci s polynomy je vhodné vytýkat nejrychleji rostoucí výraz.

$$\log(\log(n)) \ll 1000\log n \ll n^{\frac{1}{1000}} \ll n \ll n^{1000} \ll 1{,}0001^n \ll n! \ll n^n \ll (2n)!$$

R.Š

$$\lim_{n\to\infty} \frac{n}{2^n} = 0$$

R.Š

$$\lim_{n\to\infty} \frac{n!}{2^n} = \infty$$

R.Š

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$$

R.Š

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

R.Š

$$\lim_{n\to\infty} \frac{n^n}{n!} = \infty$$

$$\lim_{n\to\infty} \frac{n + 102\log n + n^2}{\sqrt{n} - n^{\frac{3}{2}} + 2^n} = \lim_{n\to\infty} \frac{n^2\left(\frac{1}{n} + \frac{102\log n}{n^2} + 1\right)}{2^n\left(\frac{\sqrt{n}}{2^n} - \frac{n^{\frac{3}{2}}}{2^n} + 1\right)}$$

$$= \overset{\text{R.Š}}{\lim_{n\to\infty} \frac{n^2}{2^n}} \lim_{n\to\infty} \frac{\frac{1}{n} + \frac{102\log n}{n^2} + 1}{\frac{\sqrt{n}}{2^n} - \frac{n^{\frac{3}{2}}}{2^n} + 1}$$

$$= 0 \times \frac{\lim\limits_{n\to\infty} \frac{1}{n} + \overset{\text{R.Š}}{\lim\limits_{n\to\infty} \frac{102\log n}{n^2}} + \lim\limits_{n\to\infty} 1}{\underset{\text{R.Š}}{\lim\limits_{n\to\infty} \frac{\sqrt{n}}{2^n}} - \underset{\text{R.Š}}{\lim\limits_{n\to\infty} \frac{n^{\frac{3}{2}}}{2^n}} + \lim\limits_{n\to\infty} 1} = 0 \times \frac{0 + 0 + 1}{0 - 0 + 1} = 0$$

$$\lim_{n\to\infty} \frac{\sqrt{4n - 2\log n + 11n^2}}{\sqrt{n} - n} = \lim_{n\to\infty} \frac{\sqrt{n^2(\frac{4}{n} - 2\frac{\log n}{n^2} + 11)}}{n(\frac{\sqrt{n}}{n} - 1)} = \lim_{n\to\infty} \frac{\cancel{n}.\sqrt{(\frac{4}{n} - 2\frac{\log n}{n^2} + 11)}}{\cancel{n}(\frac{\sqrt{n}}{n} - 1)}$$

$$= \frac{\lim_{n\to\infty} \sqrt{(\frac{4}{n} - 2\frac{\log n}{n^2} + 11)}}{\lim_{n\to\infty} (\frac{\sqrt{n}}{n} - 1)} = \frac{\sqrt{\lim_{n\to\infty} (\frac{4}{n} - 2\overset{\text{R.Š}}{\frac{\log n}{n^2}} + 11)}}{\lim_{n\to\infty} (\underset{\text{R.Š}}{\frac{\sqrt{n}}{n}} - 1)}$$

$$= \frac{\sqrt{0 - 0 + 11}}{0 - 1} = -\sqrt{11}$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{4n-2\log n+11.27^n}}{(\frac{7}{2})^n+2^n} = \lim_{n\to\infty}\frac{\sqrt[3]{27^n(\frac{4n}{27^n}-\frac{2\log n}{27^n}+11)}}{(3,5)^n+2^n}$$

$$= \lim_{n\to\infty}\frac{(3^n)(\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11}\ )}{(3,5)^n(1+\frac{2^n}{(3,5)^n})} = \lim_{n\to\infty}(\frac{3}{3,5})^n\frac{(\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11}\ )}{(1+(\frac{2}{3,5})^n)}$$

$$= \lim_{n\to\infty}(\frac{3}{3,5})^n \lim_{n\to\infty}\frac{(\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11}\ )}{(1+(\frac{2}{3,5})^n)} = \lim_{n\to\infty}(\frac{3}{3,5})^n\times\frac{\lim_{n\to\infty}\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11}}{\lim_{n\to\infty}(1+(\frac{2}{3.5})^n)}$$

$$= \lim_{n\to\infty}(\frac{3}{3,5})^n\times\frac{\sqrt[3]{\lim_{n\to\infty}(\frac{4n}{27^n}-\frac{2\log n}{27^n}+11)}}{\lim_{n\to\infty}1+\lim_{n\to\infty}(\frac{2}{3.5})^n} = 0\times\frac{\sqrt[3]{11}}{1} = 0$$

$$\lim_{n\to\infty}\frac{1^n+2^n+3^n+4^n+5^n}{(5,0001)^n} = \lim_{n\to\infty}\frac{5^n((\frac{1}{5})^n+(\frac{2}{5})^n+(\frac{3}{5})^n+(\frac{4}{5})^n+1)}{(5,0001)^n}$$

$$= \lim_{n\to\infty}\left(\frac{5}{5,0001}\right)^n((\frac{1}{5})^n+(\frac{2}{5})^n+(\frac{3}{5})^n+(\frac{4}{5})^n+1)$$

$$= \lim_{n\to\infty}\left(\frac{5}{5,0001}\right)^n \lim_{n\to\infty}((\frac{1}{5})^n+(\frac{2}{5})^n+(\frac{3}{5})^n+(\frac{4}{5})^n+1)$$

$$= 0(0+0+0+0+1) = 0$$

Věta (n-tá odmocnina).

(a) Pro $c > 0$ $\lim_{n\to\infty} \sqrt[n]{c} = 1$.

(b) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

$$\lim_{n\to\infty} \sqrt[n]{2^n + 4^n} = 4$$

$$\lim_{n\to\infty} \sqrt[n]{2^n + 4^n} = \lim_{n\to\infty} \sqrt[n]{4^n\left(\left(\frac{2}{4}\right)^n+1\right)} = \lim_{n\to\infty} 4 . \sqrt[n]{\underbrace{\left(\frac{1}{2}\right)^n}_{\to 0}+1} = 4$$

$$= 4 \cdot \sqrt[n]{\lim_{n\to\infty}\left(\left(\frac{1}{2}\right)^n+1\right)}$$

Výsledek je sice správný, ale postup je špatný, protože když je proměnná *n*, limita nemůže projít *n*-tou odmocninou.
To znamená, že nemůžeme nejdřív vypočítat limitu výrazu pod odmocninou a potom vzít *n*-tou odmocninu.

$$\lim_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{\lim_{n\to\infty} a_n} \qquad \lim_{n\to\infty} (a_n)^k = (\lim_{n\to\infty} a_n)^k$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} \neq \sqrt[n]{\lim_{n\to\infty} a_n} \qquad \lim_{n\to\infty} (a_n)^n \neq (\lim_{n\to\infty} a_n)^n$$

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \neq \left(\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{2^n + 4^n} = 4$$

$$4 = \sqrt[n]{4^n} \leq \sqrt[n]{2^n + 4^n} \leq \sqrt[n]{4^n + 4^n} = \sqrt[n]{2.4^n} = 4\sqrt[n]{2}$$

$$\lim_{n\to\infty} 4 = 4$$

$$\lim_{n\to\infty} 4\sqrt[n]{2} = 4 \lim_{n\to\infty} \sqrt[n]{2} = 4 \,.\, 1 = 4$$

Věta o dvou policajtech

$$\Rightarrow \qquad \lim_{n\to\infty} \sqrt[n]{2^n + 4^n} = 4$$

$$\lim_{n\to\infty} \sqrt[n]{n^3 + 3^n + 13^n + \ln n} = 13$$

$$13 = \sqrt[n]{13^n} \leq \sqrt[n]{n^3 + 3^n + 13^n + \ln n} \leq \sqrt[n]{13^n + 13^n + 13^n + 13^n} = \sqrt[n]{4 \times 13^n} = 13\sqrt[n]{4}$$

$$\lim_{n\to\infty} 13 = 13$$

$$\lim_{n\to\infty} 13\sqrt[n]{4} = 13 \lim_{n\to\infty} \sqrt[n]{4} = 13 \,.\, 1 = 13$$

Věta o dvou policajtech

$$\Rightarrow \quad \lim_{n\to\infty} \sqrt[n]{n^3 + 3^n + 13^n + \ln n} = 13$$

$$\lim_{n\to\infty} \sqrt[n]{2^n - n^5 + \ln n} = 2$$

$$2.\sqrt[n]{\frac{1}{2}} = \sqrt[n]{\frac{1}{2}.2^n} = \sqrt[n]{2^n - \frac{1}{2}.2^n} \leq \sqrt[n]{2^n \underbrace{- n^5 + \ln n}_{<0}} \leq \sqrt[n]{2^n} = 2$$

$$\lim_{n\to\infty} 2 = 2$$

$$\lim_{n\to\infty} 2.\sqrt[n]{\frac{1}{2}} = 2\lim_{n\to\infty}\sqrt[n]{\frac{1}{2}} = 2.1 = 2$$

Věta o dvou policajtech

$$\Rightarrow \quad \lim_{n\to\infty} \sqrt[n]{2^n - n^5 + \ln n} = 2$$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{2}\right)^n + \frac{1}{n^n} - \left(\frac{1}{3}\right)^n} = \frac{1}{2}$$

$$\frac{1}{2} \cdot \sqrt[n]{\frac{1}{2}} = \sqrt[n]{\frac{1}{2} \cdot \left(\frac{1}{2}\right)^n} = \sqrt[n]{\left(\frac{1}{2}\right)^n - \frac{1}{2}\left(\frac{1}{2}\right)^n} \leq \sqrt[n]{\left(\frac{1}{2}\right)^n + \underbrace{\frac{1}{n^n} - \left(\frac{1}{3}\right)^n}_{<\,0}} \leq \sqrt[n]{\left(\frac{1}{2}\right)^n} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{1}{2} \cdot \sqrt[n]{\frac{1}{2}} = \frac{1}{2} \lim_{n\to\infty} \sqrt[n]{\frac{1}{2}} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

Věta o dvou policajtech

$$\Rightarrow \quad \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{2}\right)^n + \frac{1}{n^n} - \left(\frac{1}{3}\right)^n} = \frac{1}{2}$$