Předmět: NMTM101 Matematická analýza I

Typ výuky: Cvičení

Vahid Borji
borji@karlin.mff.cuni.cz

Cvičení 4

Věta o aritmetice limit

(VOAL)

Jsou-li $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ **posloupnost,**

Pokud pravá strana má smysl

(i) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

(ii) $\lim\limits_{n\to\infty}(a_n . b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

(iii) $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$

$$\lim_{n\to\infty}(a_n)^m = (\lim_{n\to\infty} a_n)^m$$

$$\lim_{n\to\infty}\sqrt[k]{a_n} = \sqrt[k]{\lim_{n\to\infty} a_n}$$

Lemma („Nulová · omezená = nulová.“).

Nechť $\{a_n\}_{n=1}^{\infty}$ a $\{b_n\}_{n=1}^{\infty}$ jsou posloupnosti

a nechť $\lim_{n\to\infty} a_n = 0$ a $\{b_n\}_{n=1}^{\infty}$ je omezená.

Potom $\lim_{n\to\infty} (a_n . b_n) = 0$

Lemma (O dvou policajtech).

Nechť $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ a $\{c_n\}_{n=1}^{\infty}$ jsou posloupnosti splňující následující podmínky:

(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a \in \mathbb{R}$;

(b) $\exists n_1 \in \mathbb{N}\ \forall n \geq n_1 : a_n \leq c_n \leq b_n$.

Pak je posloupnost $\{c_n\}_{n=1}^{\infty}$ konvergentní a jest $\lim_{n\to\infty} c_n = a$.

Vypočtěte $\lim\limits_{n\to\infty} \frac{3n+5}{7n+11} \neq \frac{\lim\limits_{n\to\infty} 3n+5}{\lim\limits_{n\to\infty} 7n+11} = \frac{\infty}{\infty}$

$$\lim_{n\to\infty} \frac{3n+5}{7n+11} = \lim_{n\to\infty} \frac{\cancel{n}\,(3+\frac{5}{n})}{\cancel{n}\,(7+\frac{11}{n})} = \frac{\lim\limits_{n\to\infty}(3+\frac{5}{n})}{\lim\limits_{n\to\infty}(7+\frac{11}{n})} = \frac{\lim\limits_{n\to\infty} 3 + \lim\limits_{n\to\infty}\frac{5}{n}}{\lim\limits_{n\to\infty} 7 + \lim\limits_{n\to\infty}\frac{11}{n}}$$

$$= \frac{3 + \lim\limits_{n\to\infty} 5 \cdot \lim\limits_{n\to\infty}\frac{1}{n}}{7 + \lim\limits_{n\to\infty} 11 \cdot \lim\limits_{n\to\infty}\frac{1}{n}} = \frac{3+5.0}{7+11.0} = \frac{3}{7}$$

Vypočtěte $\lim\limits_{n\to\infty} \frac{n^2}{n^2+1} \neq \frac{\lim\limits_{n\to\infty} n^2}{\lim\limits_{n\to\infty} n^2+1} = \frac{\infty}{\infty}$

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}\left(1+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} \overset{?}{=} \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n^2}\right)} \overset{?}{=} \frac{1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n^2}}$$

$$= \frac{1}{1+0} = 1$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3}=\lim_{n\to\infty}\frac{\cancel{n^2}\left(\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\left(\frac{3}{n}\right)^2+\cdots+1\right)}{\overset{n}{\cancel{n^3}}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\ \lim_{n\to\infty}\left(\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\left(\frac{3}{n}\right)^2+\cdots+1\right)$$

$$\neq 0\times\boxed{\left[\lim_{n\to\infty}\left(\frac{1}{n}\right)^2+\lim_{n\to\infty}\left(\frac{2}{n}\right)^2+\cdots+\lim_{n\to\infty}1\right]}$$

$$=0\times[0+0+\cdots+1]=0$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3}\neq\frac{\lim\limits_{n\to\infty}(1^2+2^2+\cdots+n^2)}{\lim\limits_{n\to\infty}n^3}=\frac{\infty}{\infty}$$

$$\boxed{1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}}$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3}=\lim_{n\to\infty}\frac{\frac{n(n+1)(2n+1)}{6}}{n^3}=\lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3}$$

$$=\lim_{n\to\infty}\frac{\cancel{n^3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)}{6\cancel{n^3}}=\frac{\lim\limits_{n\to\infty}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)}{\lim\limits_{n\to\infty}6}=\cdots=\frac{2+0+0}{6}=\frac{1}{3}$$

$$\lim_{n\to\infty}\left(\frac{1+2+3+\cdots+n}{2+n}-\frac{n}{2}\right) \neq \frac{\lim_{n\to\infty}(1+2+3+\cdots+n)}{\lim_{n\to\infty}(2+n)} - \lim_{n\to\infty}\left(\frac{n}{2}\right) = \frac{\infty}{\infty} - \infty$$

$$\boxed{1+2+3+\cdots+n = \frac{n(n+1)}{2}}$$

$$\lim_{n\to\infty}\left(\frac{1+2+3+\cdots+n}{2+n}-\frac{n}{2}\right) = \lim_{n\to\infty}\left(\frac{\frac{n(n+1)}{2}}{2+n}-\frac{n}{2}\right) = \lim_{n\to\infty}\left(\frac{n^2+n}{2(2+n)}-\frac{n}{2}\right)$$

$$= \lim_{n\to\infty}\frac{n^2+n-n(2+n)}{2(2+n)} = \lim_{n\to\infty}\frac{\cancel{n^2}+n-2n-\cancel{n^2}}{2(2+n)} = \lim_{n\to\infty}\frac{-n}{4+2n} = \lim_{n\to\infty}\frac{-\cancel{n}}{\cancel{n}\left(\frac{4}{n}+2\right)}$$

$$= \lim_{n\to\infty}\frac{-1}{\frac{4}{n}+2} = \frac{\lim_{n\to\infty}(-1)}{\lim_{n\to\infty}\left(\frac{4}{n}+2\right)} = \frac{-1}{\lim_{n\to\infty}4 \,.\, \lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}2} = \frac{-1}{4\,.\,0+2} = -\frac{1}{2}$$

Nulová

Omezená

$$-1 \le \sin(n!) \le 1$$

$$\lim_{n\to\infty} \frac{\sqrt[3]{n}\,\sin(n!)}{n+1} = \lim_{n\to\infty} \frac{\sqrt[3]{n}}{n+1} \sin(n!) = 0$$

$$\lim_{n\to\infty} \frac{\sqrt[3]{n}}{n+1} = \lim_{n\to\infty} \frac{n^{\frac{1}{3}}}{n+1} = \lim_{n\to\infty} \frac{\cancel{n^{\frac{1}{3}}}}{\cancel{n^{\frac{1}{3}}}\left(n^{\frac{2}{3}} + n^{\frac{-1}{3}}\right)} = \lim_{n\to\infty} \frac{1}{n^{\frac{2}{3}} + n^{\frac{-1}{3}}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} (n)^{\frac{2}{3}} + \lim_{n\to\infty} (n)^{\frac{-1}{3}}}$$

$$= \frac{1}{\left(\lim_{n\to\infty} n\right)^{\frac{2}{3}} + \left(\lim_{n\to\infty} n\right)^{\frac{-1}{3}}} = \frac{1}{\infty + 0} = 0$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{n}\sin(n!)}{n+1}=?\qquad -1\le\sin(n!)\le 1\qquad \frac{\sqrt[3]{n}}{n+1}>0$$

$$-\frac{\sqrt[3]{n}}{n+1}\le\frac{\sqrt[3]{n}\,\sin(n!)}{n+1}\le\frac{\sqrt[3]{n}}{n+1}$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{n}}{n+1}=\lim_{n\to\infty}\frac{n^{\frac{1}{3}}}{n+1}=\lim_{n\to\infty}\frac{\cancel{n^{\frac{1}{3}}}}{\cancel{n^{\frac{1}{3}}}(n^{\frac{2}{3}}+n^{\frac{-1}{3}})}=\lim_{n\to\infty}\frac{1}{n^{\frac{2}{3}}+n^{\frac{-1}{3}}}=\cdots=\frac{1}{\infty+0}=0$$

$$\lim_{n\to\infty}\left(-\frac{\sqrt[3]{n}}{n+1}\right)=-\lim_{n\to\infty}\frac{\sqrt[3]{n}}{n+1}=0$$

Věta o dvou policajtech $\Rightarrow \lim_{n\to\infty}\frac{\sqrt[3]{n}\sin(n!)}{n+1}=0$

$$\lim_{n\to\infty}\left(\frac{\sin(n)}{n}-\sqrt[3]{n}\right)=\lim_{n\to\infty}\frac{\sin(n)}{n}-\lim_{n\to\infty}\sqrt[3]{n}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sin(n)-\lim_{n\to\infty}\sqrt[3]{n}\;=0-\sqrt[3]{\lim_{n\to\infty}n}=0-\sqrt[3]{\infty}\;=0-\infty$$

$$=-\infty$$

Nulová . Omezená = Nulová

Neexistuje

$$\lim_{n\to\infty}\frac{1}{n}\sin(n)\neq\lim_{n\to\infty}\frac{1}{n}\lim_{n\to\infty}\sin(n)$$

$$\lim_{\boldsymbol{n}\to\infty} \frac{\sin \boldsymbol{n} - \cos n^2}{\boldsymbol{n}^2 + \mathbf{17}} = \lim_{\boldsymbol{n}\to\infty} \left(\frac{\mathbf{1}}{\boldsymbol{n}^2 + \mathbf{17}}\right)(\sin \boldsymbol{n} - \cos n^2)$$

Nulová . Omezená = Nulová

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) \neq \lim_{n\to\infty}\sqrt{n+1} - \lim_{n\to\infty}\sqrt{n} = \infty - \infty$$

$$\lim_{n\to\infty}\left(\underbrace{\sqrt{n+1}}_{A}-\underbrace{\sqrt{n}}_{B}\right)\frac{\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)} = \lim_{n\to\infty}\frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty}\frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}\sqrt{n+1}+\lim_{n\to\infty}\sqrt{n}}$$

$$(A-B)(A+B) = A^2 - B^2$$

$$= \frac{1}{\infty + \infty} = 0$$

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\sqrt{n}$$

$(\infty - \infty)\infty$

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\sqrt{n} = \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})}(\sqrt{n})$$

$$= \lim_{n\to\infty}\frac{(n+1-n)}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty}\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}(\frac{\sqrt{n+1}}{\sqrt{n}}+1)} = \lim_{n\to\infty}\frac{1}{\sqrt{\frac{n+1}{n}}+1} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1} = \cdots = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

$$\sqrt[3]{n^3+2n^2}=A\ ,\ \sqrt[3]{n^3-n^2+1}=B \qquad (A-B)\left(A^2+AB+B^2\right)=A^3-B^3$$

$$\lim_{n\to\infty}\left(\sqrt[3]{n^3+2n^2}-\sqrt[3]{n^3-n^2+1}\right)\frac{\left(\left(\sqrt[3]{n^3+2n^2}\right)^2+\left(\sqrt[3]{n^3+2n^2}\right)\left(\sqrt[3]{n^3-n^2+1}\right)+\left(\sqrt[3]{n^3-n^2+1}\right)^2\right)}{\left(\left(\sqrt[3]{n^3+2n^2}\right)^2+\left(\sqrt[3]{n^3+2n^2}\right)\left(\sqrt[3]{n^3-n^2+1}\right)+\left(\sqrt[3]{n^3-n^2+1}\right)^2\right)}$$

$$\lim_{n\to\infty}\frac{\cancel{n^3+2n^2-(n^3-n^2+1)}\quad 3n^2-1}{\cancel{(n^3+2n^2)^{\frac{2}{3}}}+\cancel{(n^3+2n^2)^{\frac{1}{3}}}\cancel{(n^3-n^2+1)^{\frac{1}{3}}}+\cancel{(n^3-n^2+1)^{\frac{2}{3}}}}$$

$$n^2\left(1+\frac{2}{n}\right)^{\frac{2}{3}}\quad +n\left(1+\frac{2}{n}\right)^{\frac{1}{3}}n\left(1-\frac{1}{n}+\frac{1}{n^3}\right)^{\frac{1}{3}}\ +n^2\left(1-\frac{1}{n}+\frac{1}{n^3}\right)^{\frac{2}{3}}$$

$$\lim_{n\to\infty}\frac{\cancel{n^2}\left(3-\frac{1}{n^2}\right)}{\cancel{n^2}\left[\left(1+\frac{2}{n}\right)^{\frac{2}{3}}+\left(1+\frac{2}{n}\right)^{\frac{1}{3}}\left(1-\frac{1}{n}+\frac{1}{n^3}\right)^{\frac{1}{3}}+\left(1-\frac{1}{n}+\frac{1}{n^3}\right)^{\frac{2}{3}}\right]}=\frac{3}{1+1+1}=1$$

$$\lim_{n\to\infty} \frac{\sqrt[3]{n^2+11}-\sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6}-\sqrt[3]{n^2}}$$

$$\lim_{n\to\infty} \frac{\sqrt[6]{7n^6+5n^5-7}-\sqrt[5]{2n^5+18}}{\sqrt[3]{8n^3+20n+16}-\sqrt{n^2+2}}$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{n^2+11}-\sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6}-\sqrt[3]{n^2}}\cdot\frac{\left(\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2\right)}{\left(\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2\right)}$$

$$\frac{\left(\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2\right)}{\left(\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2\right)}$$

$$\lim_{n\to\infty}\frac{\overset{10}{\left[\cancel{\left(\sqrt[3]{n^2+11}\right)^3-\left(\sqrt[3]{n^2+1}\right)^3}\right]}\left(\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2\right)}{\underset{6}{\left[\cancel{\left(\sqrt[3]{n^2+6}\right)^3-\left(\sqrt[3]{n^2}\right)^3}\right]}\left(\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2\right)}$$

$$\frac{10}{6}\lim_{n\to\infty}\frac{\overset{n^{\frac{4}{3}}(1+\frac{6}{n^2})^{\frac{2}{3}}+n^{\frac{2}{3}}(1+\frac{6}{n^2})^{\frac{1}{3}}n^{\frac{2}{3}}+n^{\frac{4}{3}}}{\cancel{(n^2+6)^{\frac{2}{3}}}+\cancel{(n^2+6)^{\frac{1}{3}}}\cancel{(n^2)^{\frac{1}{3}}}+\cancel{(n^2)^{\frac{2}{3}}}}}{(n^2+11)^{\frac{2}{3}}+(n^2+11)^{\frac{1}{3}}(n^2+1)^{\frac{1}{3}}+(n^2+1)^{\frac{2}{3}}}=\frac{5}{3}\lim_{n\to\infty}\frac{\cancel{n^{\frac{4}{3}}}\left[\left(1+\frac{6}{n^2}\right)^{\frac{2}{3}}+\left(1+\frac{6}{n^2}\right)^{\frac{1}{3}}+1\right]}{\cancel{n^{\frac{4}{3}}}\left[\left(1+\frac{11}{n^2}\right)^{\frac{2}{3}}+\left(1+\frac{11}{n^2}\right)^{\frac{1}{3}}\left(1+\frac{1}{n^2}\right)^{\frac{1}{3}}+\left(1+\frac{1}{n^2}\right)^{\frac{2}{3}}\right]}=\frac{5}{3}$$

$$\lim_{n\to\infty} \frac{\sqrt[3]{n^2+11} - \sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6} - \sqrt[3]{n^2}}$$

$$\vdots$$

$$= \frac{1 \; -1}{1 \; -1}$$

$$= \frac{0}{0}$$

$$\lim_{n\to\infty} \frac{\sqrt[6]{7n^6+5n^5-7} - \sqrt[5]{2n^5+18}}{\sqrt[3]{8n^3+20n+16} - \sqrt{n^2+2}}$$

$$= \lim_{n\to\infty} \frac{n\,(7+\frac{5}{n}-\frac{7}{n^6})^{\frac{1}{6}} - n\,(2+\frac{18}{n^5})^{\frac{1}{5}}}{n\,(8+\frac{20}{n^2}+\frac{16}{n^3})^{\frac{1}{3}} - n\,(1+\frac{2}{n^2})^{\frac{1}{2}}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}[\,(7+\frac{5}{n}-\frac{7}{n^6})^{\frac{1}{6}} - (2+\frac{18}{n^5})^{\frac{1}{5}}]}{\cancel{n}[(8+\frac{20}{n^2}+\frac{16}{n^3})^{\frac{1}{3}} - (1+\frac{2}{n^2})^{\frac{1}{2}}]}$$

$$= \frac{(7)^{\frac{1}{6}} - (2)^{\frac{1}{5}}}{(8)^{\frac{1}{3}} \; -1}$$