Předmět: NMTM101 Matematická analýza I

Typ výuky: Cvičení

Vahid Borji
borji@karlin.mff.cuni.cz

Cvičení 4

Věta o aritmetice limit

(VOAL)

Jsou-li $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ **posloupnost,**

(i) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

Pokud pravá strana má smysl

(ii) $\lim_{n\to\infty}(a_n . b_n) = \lim_{n\to\infty} a_n \,.\, \lim_{n\to\infty} b_n$

(iii) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

$$\lim_{n\to\infty}(a_n)^m = \left(\lim_{n\to\infty} a_n\right)^m$$

$$\lim_{n\to\infty}\sqrt[k]{a_n} = \sqrt[k]{\lim_{n\to\infty} a_n}$$

Lemma („Nulová · omezená = nulová.“).

Nechť $\{a_n\}_{n=1}^{\infty}$ a $\{b_n\}_{n=1}^{\infty}$ jsou posloupnosti

a nechť $\lim_{n\to\infty} a_n = 0$ a $\{b_n\}_{n=1}^{\infty}$ je omezená.

Potom $\lim_{n\to\infty} (a_n . b_n) = 0$

Lemma (O dvou policajtech).

Nechť $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ a $\{c_n\}_{n=1}^{\infty}$ jsou posloupnosti splňující následující podmínky:

(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a \in \mathbb{R}$;

(b) $\exists n_1 \in \mathbb{N}\ \forall n \geq n_1 : a_n \leq c_n \leq b_n$.

Pak je posloupnost $\{c_n\}_{n=1}^{\infty}$ konvergentní a jest $\lim_{n\to\infty} c_n = a$.

$$\lim_{n\to\infty} \frac{\sqrt[3]{n}\sin(n!)}{n+1} = \lim_{n\to\infty} \frac{\sqrt[3]{n}}{n+1}\sin(n!) = 0$$

Nulová

$$\lim_{n\to\infty} \frac{\sqrt[3]{n}}{n+1} = 0$$

Omezená

$$\sin(n!) \qquad -1 < \sin(n!) < 1$$

$$\lim_{n\to\infty} \frac{\sqrt[3]{n}\sin(n!)}{n+1} = ?$$

$$-1 < \sin(n!) < 1$$

$$\frac{\sqrt[3]{n}}{n+1} > 0$$

$$-\frac{\sqrt[3]{n}}{n+1} < \frac{\sqrt[3]{n}\,\sin(n!)}{n+1} < \frac{\sqrt[3]{n}}{n+1}$$

$$\lim_{n\to\infty} \frac{\sqrt[3]{n}}{n+1} = 0$$

Věta o dvou policajtech $\Rightarrow \lim_{n\to\infty} \frac{\sqrt[3]{n}\sin(n!)}{n+1} = 0$

$$\lim_{n\to\infty} \left(-\frac{\sqrt[3]{n}}{n+1}\right) = 0$$

$$\lim_{n\to\infty}\left(\frac{\sin(n)}{n}-\sqrt[3]{n}\right)=\lim_{n\to\infty}\frac{\sin(n)}{n}-\lim_{n\to\infty}\sqrt[3]{n}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sin(n)-\lim_{n\to\infty}\sqrt[3]{n}\quad=0-\sqrt[3]{\lim_{n\to\infty}n}=0-\sqrt[3]{\infty}=0-\infty$$

$$=-\infty$$

Nulová . Omezená = Nulová

Neexistuje

$$\lim_{n\to\infty}\frac{1}{n}\sin(n)\neq\lim_{n\to\infty}\frac{1}{n}\lim_{n\to\infty}\sin(n)$$

$$\lim_{n\to\infty} \frac{\sin \boldsymbol{n} - \cos n^2}{\boldsymbol{n^2 + 17}} = \lim_{n\to\infty} \left(\frac{\mathbf{1}}{\boldsymbol{n^2 + 17}}\right)(\sin \boldsymbol{n} - \cos n^2)$$

Nulová . Omezená = Nulová

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) \neq \lim_{n\to\infty}\sqrt{n+1} - \lim_{n\to\infty}\sqrt{n} \quad = \infty - \infty$$

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})} = \lim_{n\to\infty}\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$$

A B

$$(A-B)(A+B) = A^2 - B^2$$

$$= \lim_{n\to\infty}\frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}\sqrt{n+1} + \lim_{n\to\infty}\sqrt{n}}$$

$$= \frac{1}{\infty + \infty} = 0$$

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\sqrt{n} \qquad (\infty-\infty)\infty$$

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\sqrt{n} = \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\,\frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})}(\sqrt{n})$$

$$= \lim_{n\to\infty}\frac{(n+1-n)}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty}\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}(\frac{\sqrt{n+1}}{\sqrt{n}}+1)} = \lim_{n\to\infty}\frac{1}{\sqrt{\frac{n+1}{n}}+1} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1} = \cdots = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

$\sqrt[3]{n^3+2n^2}=A\ ,\ \sqrt[3]{n^3-n^2+1}=B$ $\qquad (A-B)\left(A^2+AB+B^2\right)=A^3-B^3$

$$\lim_{n\to\infty}\left(\sqrt[3]{n^3+2n^2}-\sqrt[3]{n^3-n^2+1}\right)\frac{\left(\left(\sqrt[3]{n^3+2n^2}\right)^2+\left(\sqrt[3]{n^3+2n^2}\right)\left(\sqrt[3]{n^3-n^2+1}\right)+\left(\sqrt[3]{n^3-n^2+1}\right)^2\right)}{\left(\left(\sqrt[3]{n^3+2n^2}\right)^2+\left(\sqrt[3]{n^3+2n^2}\right)\left(\sqrt[3]{n^3-n^2+1}\right)+\left(\sqrt[3]{n^3-n^2+1}\right)^2\right)}$$

$$\lim_{n\to\infty}\frac{n^3+2n^2-(n^3-n^2+1)\quad 3n^2-1}{\left(n^3+2n^2\right)^{\frac{2}{3}}+\left(n^3+2n^2\right)^{\frac{1}{3}}\left(n^3-n^2+1\right)^{\frac{1}{3}}+\left(n^3-n^2+1\right)^{\frac{2}{3}}}$$

$$n^2\left(1+\frac{2}{n}\right)^{\frac{2}{3}}\quad +n\left(1+\frac{2}{n}\right)^{\frac{1}{3}}\ n\left(1-\frac{1}{n}+\frac{1}{n^3}\right)^{\frac{1}{3}}\ +n^2\left(1-\frac{1}{n}+\frac{1}{n^3}\right)^{\frac{2}{3}}$$

$$\lim_{n\to\infty}\frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left[\left(1+\frac{2}{n}\right)^{\frac{2}{3}}+\left(1+\frac{2}{n}\right)^{\frac{1}{3}}\left(1-\frac{1}{n}+\frac{1}{n^3}\right)^{\frac{1}{3}}+\left(1-\frac{1}{n}+\frac{1}{n^3}\right)^{\frac{2}{3}}\right]}=\frac{3}{1+1+1}=1$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{n^2+11}-\sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6}-\sqrt[3]{n^2}}\cdot\frac{\left(\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2\right)}{\left(\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2\right)}\cdot\frac{\left(\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2\right)}{\left(\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2\right)}$$

$$\lim_{n\to\infty}\frac{\left[\left(\sqrt[3]{n^2+11}\right)^3-\left(\sqrt[3]{n^2+1}\right)^3\right]\left(\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2\right)}{\left[\left(\sqrt[3]{n^2+6}\right)^3-\left(\sqrt[3]{n^2}\right)^3\right]\left(\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2\right)}$$

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$$\frac{10}{6}\lim_{n\to\infty}\frac{n^{\frac{4}{3}}\left(1+\frac{6}{n^2}\right)^{\frac{2}{3}}+n^{\frac{2}{3}}\left(1+\frac{6}{n^2}\right)^{\frac{1}{3}}n^{\frac{2}{3}}+n^{\frac{4}{3}}}{\left(n^2+6\right)^{\frac{2}{3}}+\left(n^2+6\right)^{\frac{1}{3}}\left(n^2\right)^{\frac{1}{3}}+\left(n^2\right)^{\frac{2}{3}}}=\frac{5}{3}\lim_{n\to\infty}\frac{n^{\frac{4}{3}}\left[\left(1+\frac{6}{n^2}\right)^{\frac{2}{3}}+\left(1+\frac{6}{n^2}\right)^{\frac{1}{3}}+1\right]}{n^{\frac{4}{3}}\left[\left(1+\frac{11}{n^2}\right)^{\frac{2}{3}}+\left(1+\frac{11}{n^2}\right)^{\frac{1}{3}}\left(1+\frac{1}{n^2}\right)^{\frac{1}{3}}+\left(1+\frac{1}{n^2}\right)^{\frac{2}{3}}\right]}=\frac{5}{3}$$

$$(a+b)^{50} = \binom{50}{0} a^{50} + \binom{50}{1} a^{49} b^{1} + \binom{50}{2} a^{48} b^{2} + \cdots + \binom{50}{50} b^{50}$$

$$\lim_{n\to\infty} \frac{(n+1)^{50} - (n-1)^{50}}{n^{49}} = \lim_{n\to\infty} \frac{(n^{50} + 50n^{49} + Bn^{48} \ldots + 1) - (n^{50} - 50n^{49} + Xn^{48} \ldots + 1)}{n^{49}}$$

$$= \lim_{n\to\infty} \frac{100n^{49} + (B-X)n^{48} + \ldots + ?n}{n^{49}} = \lim_{n\to\infty} \frac{n^{49}\left(100 + \frac{(B-X)}{n} \ldots + \frac{?}{n^{48}}\right)}{n^{49}} = 100$$

Věta (Růstová škála). Od jisté hodnoty n platí:

$$\log(\log(\boldsymbol{n})) \ll \mathbf{1000}\log \boldsymbol{n} \ll \boldsymbol{n}^{\frac{\mathbf{1}}{\mathbf{1000}}} \ll \boldsymbol{n} \ll \boldsymbol{n}^{\mathbf{1000}} \ll \mathbf{1,0001}^{\boldsymbol{n}} \ll \boldsymbol{n}! \ll \boldsymbol{n}^{\boldsymbol{n}}$$

Kde zápis $a_n \ll b_n$ znamená „a_n roste výrazně pomaleji než b_n”, tedy:

$$\lim_{\boldsymbol{n}\to\infty} \frac{\boldsymbol{a_n}}{\boldsymbol{b_n}} = \mathbf{0} \qquad \lim_{\boldsymbol{n}\to\infty} \frac{\boldsymbol{b_n}}{\boldsymbol{a_n}} = \infty$$

Poučka. Stejně jako při práci s polynomy je vhodné vytýkat nejrychleji rostoucí výraz.

$$\lim_{n\to\infty} \frac{n}{2^n} = ?$$

$$\log(\log(n)) \ll 1000 \log n \ll n^{\frac{1}{1000}} \ll n \ll n^{1000} \ll 1,0001^n \ll n! \ll n^n$$

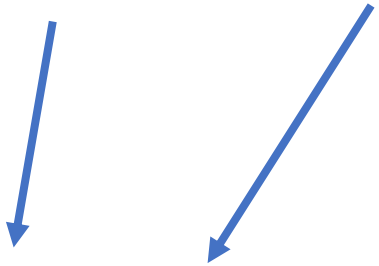


$$n \ll 2^n$$

$$\lim_{n\to\infty} \frac{n}{2^n} = 0$$

$$\lim_{n\to\infty} \frac{2^n}{n!}$$

$$\log(\log(n)) \ll 1000 \log n \ll n \ll n^{1000} \ll 1{,}0001^n \ll n! \ll n^n$$



$$2^n \ll n!$$

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0$$

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}$$

$$\log(\log(n)) \ll 1000 \log n \ll n^{\frac{1}{1000}} \ll n \ll n^{1000} \ll 1,0001^n \ll n! \ll n^n$$

$$\ln n \ll n^{\frac{1}{2}} = \sqrt{n}$$

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$$

$$\lim_{n\to\infty} \frac{n!}{n^n}$$

$$\log(\log(n)) \ll 1000 \log n \ll n^{\frac{1}{1000}} \ll n \ll n^{1000} \ll 1,0001^n \ll n! \ll n^n$$

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

$$\lim_{n\to\infty} \frac{n^n}{n!} = \infty$$

$$\lim_{n\to\infty}\frac{n+102\log n+n^2}{\sqrt{n}-n^{\frac{3}{2}}+2^n}=\lim_{n\to\infty}\frac{n^2\left(\frac{1}{n}+\frac{102\log n}{n^2}+1\right)}{2^n\left(\frac{\sqrt{n}}{2^n}-\frac{n^{\frac{3}{2}}}{2^n}+1\right)}$$

$$=\lim_{n\to\infty}\frac{n^2}{2^n}\lim_{n\to\infty}\frac{\frac{1}{n}+\frac{102\log n}{n^2}+1}{\frac{\sqrt{n}}{2^n}-\frac{n^{\frac{3}{2}}}{2^n}+1}$$

$$=0\times\frac{0+0+1}{0-0+1}=0$$

$$\lim_{n\to\infty} \frac{\sqrt{4n - 2\log n + 11n^2}}{\sqrt{n} - n} = \lim_{n\to\infty} \frac{\sqrt{n^2\left(\frac{4}{n} - 2\frac{\log n}{n^2} + 11\right)}}{n\left(\frac{\sqrt{n}}{n} - 1\right)} = \lim_{n\to\infty} \frac{n \times \sqrt{\left(\frac{4}{n} - 2\frac{\log n}{n^2} + 11\right)}}{n\left(\frac{\sqrt{n}}{n} - 1\right)}$$

$$= \lim_{n\to\infty} \frac{\sqrt{\left(\frac{4}{n} - 2\frac{\log n}{n^2} + 11\right)}}{\left(\frac{\sqrt{n}}{n} - 1\right)} = \frac{\lim_{n\to\infty} \sqrt{\left(\frac{4}{n} - 2\frac{\log n}{n^2} + 11\right)}}{\lim_{n\to\infty} \left(\frac{\sqrt{n}}{n} - 1\right)} = \frac{\sqrt{\lim_{n\to\infty} \left(\frac{4}{n} - 2\frac{\log n}{n^2} + 11\right)}}{\lim_{n\to\infty} \left(\frac{\sqrt{n}}{n} - 1\right)}$$

Pro $k \in \mathbb{N}$ platí:

$$\lim_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{\lim_{n\to\infty} a_n}$$

$$\lim_{n\to\infty} {a_n}^k = \left(\lim_{n\to\infty} a_n\right)^k$$

$$= \frac{\sqrt{0 - 0 + 11}}{0 - 1} = -\sqrt{11}$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{4n-2\log n+11.27^n}}{(\frac{7}{2})^n+2^n}=\lim_{n\to\infty}\frac{\sqrt[3]{27^n(\frac{4n}{27^n}-\frac{2\log n}{27^n}+11)}}{(3,5)^n+2^n}$$

$$=\lim_{n\to\infty}\frac{(3^n)(\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11})}{(3,5)^n(1+\frac{2^n}{(3,5)^n})}=\lim_{n\to\infty}(\frac{3}{3,5})^n\frac{(\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11})}{(1+(\frac{2}{3,5})^n)}$$

$$=\lim_{n\to\infty}(\frac{3}{3,5})^n\lim_{n\to\infty}\frac{(\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11})}{(1+(\frac{2}{3,5})^n)}=\lim_{n\to\infty}(\frac{3}{3,5})^n\times\frac{\lim_{n\to\infty}\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11}}{\lim_{n\to\infty}(1+(\frac{2}{3.5})^n)}$$

$$=\lim_{n\to\infty}(\frac{3}{3,5})^n\times\frac{\sqrt[3]{\lim_{n\to\infty}(\frac{4n}{27^n}-\frac{2\log n}{27^n}+11)}}{\lim_{n\to\infty}1+\lim_{n\to\infty}(\frac{2}{3.5})^n}=0\times\frac{\sqrt[3]{11}}{1}=0$$

$$\lim_{n\to\infty}\frac{1^n+2^n+3^n+4^n+5^n}{(5.0001)^n}=\lim_{n\to\infty}\frac{5^n((\frac{1}{5})^n+(\frac{2}{5})^n+(\frac{3}{5})^n+(\frac{4}{5})^n+1)}{(5.0001)^n}$$

$$=\lim_{n\to\infty}\left(\frac{5}{5.0001}\right)^n((\frac{1}{5})^n+(\frac{2}{5})^n+(\frac{3}{5})^n+(\frac{4}{5})^n+1)$$

$$=\lim_{n\to\infty}\left(\frac{5}{5.0001}\right)^n\lim_{n\to\infty}((\frac{1}{5})^n+(\frac{2}{5})^n+(\frac{3}{5})^n+(\frac{4}{5})^n+1)$$

$$=0(0+0+0+0+1)\ =0$$

$$\lim_{n\to\infty}\frac{n^2-13.2^n-\ln n+\sin n}{n^{97}+2^n-e^{n+2}}$$

Nulová . Omezená = Nulová

$$=\lim_{n\to\infty}\frac{2^n\left(\frac{n^2}{2^n}-13-\frac{\ln n}{2^n}+\frac{\sin n}{2^n}\right)}{e^n\left(\frac{n^{97}}{e^n}+\frac{2^n}{e^n}-e^2\right)}=0\times\frac{-13}{-e^2}=0$$

$\frac{n^2}{2^n}\to 0$, $\frac{\ln n}{2^n}\to 0$, $\frac{\sin n}{2^n}\to 0$, $2^n\to 0$

$\frac{n^{97}}{e^n}\to 0$, $\frac{2^n}{e^n}\to 0$

R.Š

$$\lim_{n\to\infty}\frac{n^2.3^n}{4^n}=\lim_{n\to\infty}n^2\left(\frac{3}{4}\right)^n=\lim_{n\to\infty}n^2\frac{1}{\left(\frac{4}{3}\right)^n}=\lim_{n\to\infty}\frac{n^2}{\left(\frac{4}{3}\right)^n}=0$$

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$$\lim_{n\to\infty}\frac{n^{1000}.\ln n}{2^n}=\lim_{n\to\infty}\frac{n^{1000}.\ln n}{(\sqrt{2})^n(\sqrt{2})^n}=0$$

$\to 0$ $\to 0$