Předmět: NMTM101 Matematická analýza I

Typ výuky: Cvičení

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Cvičení 3

Znegujte následující výroky a rozhodněte, jestli platí výrok, nebo jeho negace:

$\boldsymbol{a})$ $\forall \boldsymbol{x} \in \mathbb{R}\ \forall \boldsymbol{y} \in \mathbb{R}\colon\ \boldsymbol{x^2 + y^2 > 0}$ **Výrok neplatí**

Negace:

$\neg\ (\forall \boldsymbol{x} \in \mathbb{R}\ \forall \boldsymbol{y} \in \mathbb{R}\colon\ \boldsymbol{x^2 + y^2 > 0})$

$\exists \boldsymbol{x} \in \mathbb{R}\ \exists \boldsymbol{y} \in \mathbb{R}\colon\ \boldsymbol{x^2 + y^2 \leq 0}$ $\boldsymbol{x = y = 0}\colon\ \boldsymbol{x^2 + y^2 = 0 \leq 0}$

Negace platí

$\boldsymbol{b})\quad \forall \boldsymbol{x} \in \mathbb{R}\ \exists \boldsymbol{y} \in \mathbb{N}: (\boldsymbol{y} \leq \boldsymbol{x})\ \wedge (\boldsymbol{y} + \mathbf{1} > \boldsymbol{x})$

Výrok neplatí

Negace:

$$\neg(\ \forall \boldsymbol{x} \in \mathbb{R}\ \exists \boldsymbol{y} \in \mathbb{N}: (\boldsymbol{y} \leq \boldsymbol{x})\ \wedge (\boldsymbol{y} + \mathbf{1} > \boldsymbol{x})\ )$$

$$\exists \boldsymbol{x} \in \mathbb{R}\ \ \forall \boldsymbol{y} \in \mathbb{N}:\ (\boldsymbol{y} > \boldsymbol{x})\ \vee (\boldsymbol{y} + \mathbf{1} \leq \boldsymbol{x})$$

$\boldsymbol{x} = \mathbf{0}$: $\quad \boldsymbol{y} > \mathbf{0} \quad \vee \quad \boldsymbol{y} \leq -\mathbf{1}$

Negace platí

Nalezněte supremum, infimum, maximum a minimum následujících množin.

$A = [2, \pi)$

$B = \mathbb{N}$

$C = \{3, \quad 3.1, \quad \sqrt{3}, \quad \ln(2), \quad \pi, \quad e, \quad \frac{22}{7}\}$

$D = (\pi, \frac{22}{7}] \cap \mathbb{Q}$

$M = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$

$X = \{\frac{n}{n+1} : n \in \mathbb{N}\}$

$Y = \{1 - \frac{n}{n+1} : n \in \mathbb{N}\}$

$Z = \{x < \frac{1}{x} : x \in \mathbb{R}\}$

$W = \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$

$A = [2, \pi)$

$inf(A) = 2$

$min(A) = 2$

$sup(A) = \pi$

$max(A)$ ✗

$\boldsymbol{B = \mathbb{N}}$

$$\boldsymbol{B = \{1, 2, 3, 4, 5, \ldots\}}$$

$$\boldsymbol{inf(B) = 1} \qquad\qquad \boldsymbol{sup(B) = \infty}$$

$$\boldsymbol{min(B) = 1} \qquad\qquad \boldsymbol{max(B)} \ ❌$$

$$C = \{3, \quad 3.1, \quad \sqrt{3}, \quad \ln(2), \quad \pi, \quad e, \quad \frac{22}{7}\}$$

$$C = \{3, \quad 3.1, \quad 1.732, \quad 0.693, \quad 3.1415, \quad 2.71, \quad 3.1428\}$$

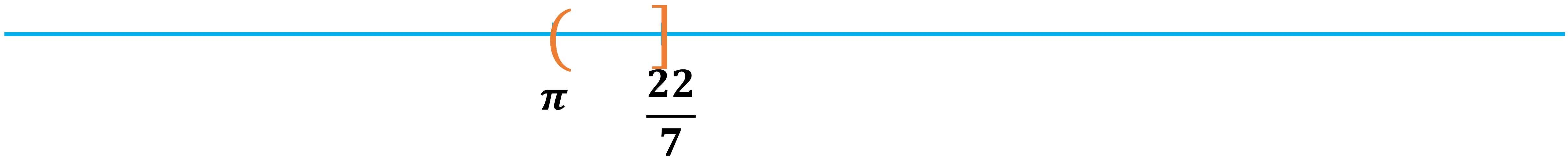
$$inf(C) = \ln(2)$$

$$min(C) = \ln(2)$$

$$sup(C) = \frac{22}{7}$$

$$max(C) = \frac{22}{7}$$

$$D = \left(\pi, \frac{22}{7}\right] \cap \mathbb{Q}$$



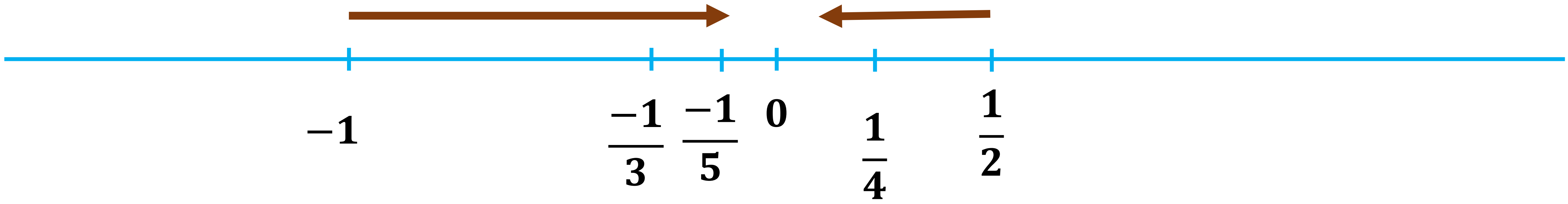
$$inf(D) = \pi \qquad sup(D) = \frac{22}{7}$$

$$min(D) =$$ 

$$max(D) = \frac{22}{7}$$

$$M = \left\{\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$$

$n \in \mathbb{N}$	1	2	3	4	5
$\frac{(-1)^n}{n}$	-1	$\frac{1}{2}$	$\frac{-1}{3}$	$\frac{1}{4}$	$\frac{-1}{5}$

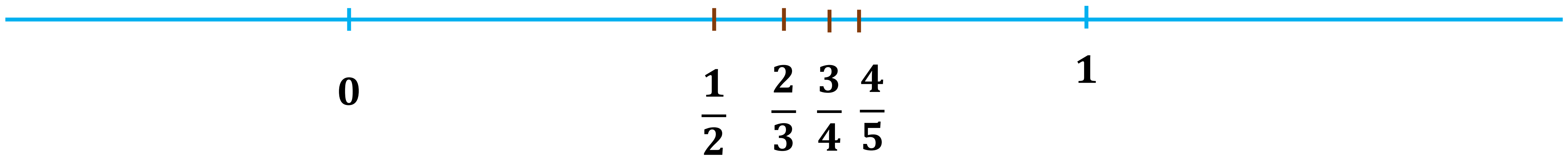


$$inf(M) = -1 \qquad sup(M) = \frac{1}{2}$$

$$min(M) = -1 \qquad max(M) = \frac{1}{2}$$

$$X = \{\frac{n}{n+1} : n \in \mathbb{N}\}$$

$n \in \mathbb{N}$	1	2	3	4	5	$\longrightarrow$	n
$\frac{n}{n+1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\longrightarrow$	$\frac{n}{n+1} < 1$

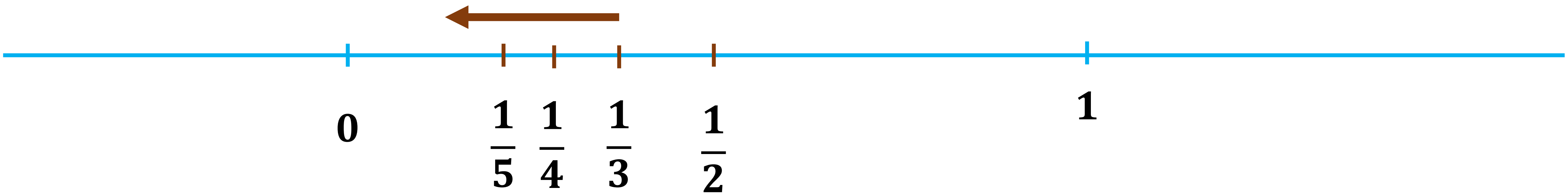


$$inf(X) = \frac{1}{2} \qquad sup(X) = 1$$

$$min(X) = \frac{1}{2} \qquad max(X) =$$

$$Y = \{1 - \frac{n}{n+1} : n \in \mathbb{N}\} = \{\frac{1}{n+1} : n \in \mathbb{N}\}$$

$n \in \mathbb{N}$	1	2	3	4	5	$\longrightarrow$	n
$\frac{1}{n+1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\longrightarrow$	$\frac{1}{n+1} > 0$



$inf(Y) = 0$ $\qquad$ $sup(Y) = \frac{1}{2}$

$min(Y) =$ $\qquad$ $max(Y) = \frac{1}{2}$

$$Z = \{x < \frac{1}{x} : x \in \mathbb{R}\}$$

$$x^2 - 1 = 0 \Rightarrow x = 1 \vee x = -1$$

$$x < \frac{1}{x} \iff x - \frac{1}{x} < 0 \iff \frac{x^2-1}{x} < 0$$

$$x = 0$$

	$(-\infty, -1)$		$[-1, 0)$		$[0, 1)$		$[1, \infty)$
$x^2 - 1$	$+$	0	$-$		$-$	0	$+$
x	$-$		$-$	0	$+$		$+$
$\frac{x^2-1}{x}$	$-$	0	$+$	∞	$-$	0	$+$

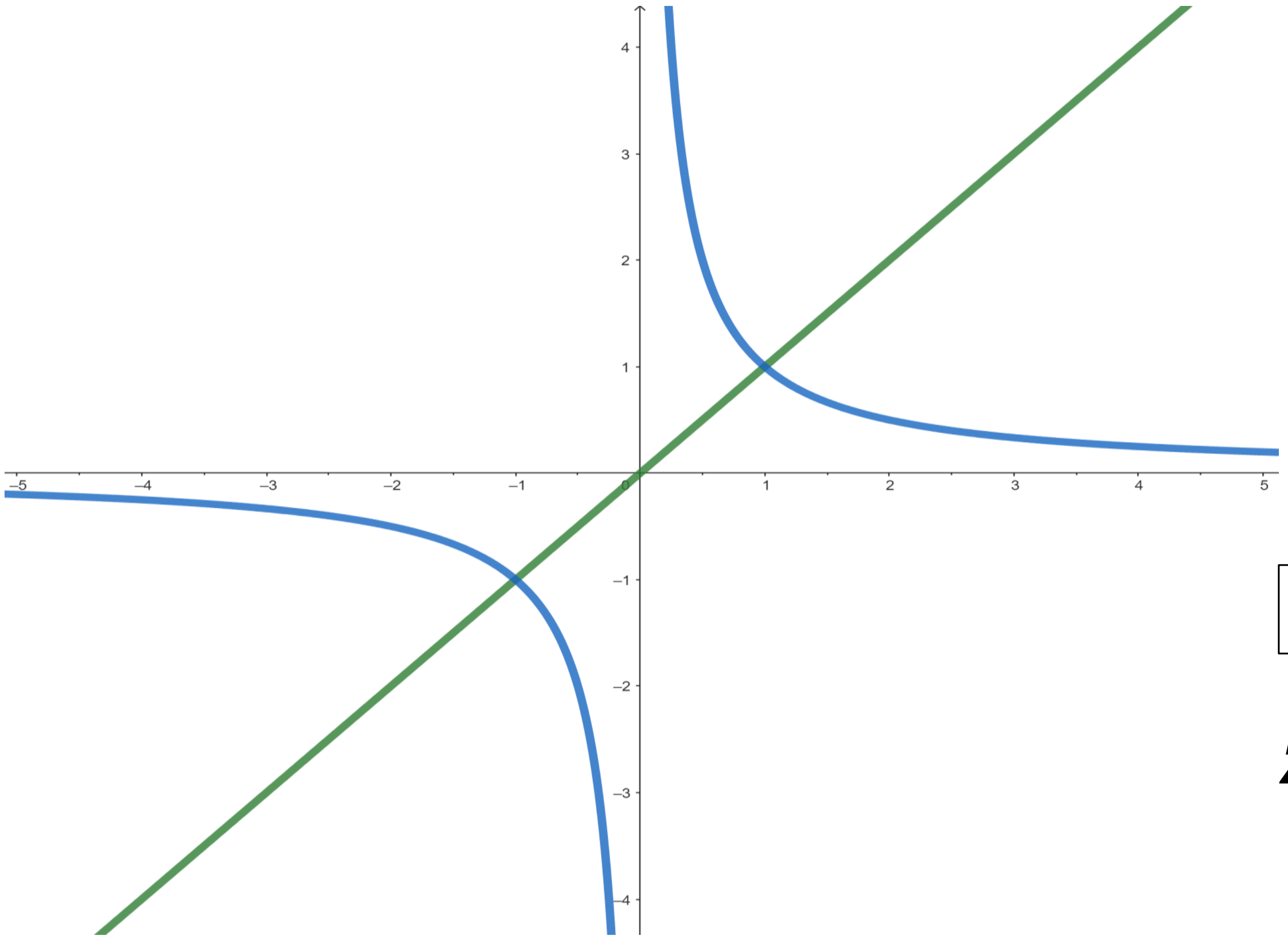
$$Z = (-\infty, -1) \cup (0, 1)$$

$$\boxed{x \in (-\infty, -1) \cup (0, 1)}$$

$$Z = \{x < \frac{1}{x} : x \in \mathbb{R}\}$$

$$f(x) = x$$

$$g(x) = \frac{1}{x}$$



$$\boxed{x \in (-\infty, -1) \cup (0, 1)}$$

$$Z = (-\infty, -1) \cup (0, 1)$$

$$Z = \{x < \frac{1}{x} : x \in \mathbb{R}\}$$

$$Z = (-\infty, -1) \cup (0, 1)$$

$inf(Z) = \ -\infty$

$sup(Z) = \ 1$

$min(Z)$

$max(Z)$

$$W = \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$$

$$\frac{1}{n} - \frac{1}{m}$$

$$n = 1: \quad \frac{1}{n} - \frac{1}{m} = 1 - \frac{1}{m} < 1$$

$$\frac{1}{n} - \frac{1}{m}$$

$$m = 1: \quad \frac{1}{n} - \frac{1}{m} = \frac{1}{n} - 1 > -1$$

$$-1 < \frac{1}{n} - \frac{1}{m} < 1$$

$sup(W) = 1$ $\quad max(W)$ ✗

$inf(W) = -1$ $\quad min(W)$ ✗

Uvažujme libovolnou množinu $\boldsymbol{M}$. Platí nutně následující nerovnost?

$$\boldsymbol{supM \geq infM}$$

$$\boldsymbol{M = \emptyset}$$

$$\boldsymbol{sup\emptyset = -\infty}$$

$$\boldsymbol{inf\emptyset = \infty}$$

LIMITA POSLOUPNOSTI

Definice (Vlastní limita posloupnosti)**.** Nechť $\{a_n\}_{n=1}^{\infty}$ je posloupnost. Řekneme, že číslo A je limita posloupnosti $\{a_n\}_{n=1}^{\infty}$ a píšeme $\lim_{n\to\infty} a_n = A$ jestliže platí:

$$\forall \boldsymbol{\varepsilon} > \mathbf{0}\ \exists \boldsymbol{n_0} \in \mathbb{N}\ \forall \boldsymbol{n} \geq \boldsymbol{n_0}\colon\ |\boldsymbol{a_n} - \boldsymbol{A}| < \boldsymbol{\varepsilon}$$

Podle definice limity posloupnosti ověřte následující rovnost.

$$\lim_{n\to\infty} a_n = A \quad \Leftrightarrow \quad \forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 \colon\ |a_n - A| < \varepsilon$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty}\frac{1}{n}=0$$

Podle definice limity posloupnosti

$$\forall\varepsilon>0\ \exists n_0\in\mathbb{N}\ \forall n\geq n_0:\ \left|\frac{1}{n}-0\right|<\varepsilon$$

$$\varepsilon>0:\qquad \left|\frac{1}{n}-0\right|<\varepsilon\quad\Leftrightarrow\frac{1}{n}<\varepsilon\quad\Leftrightarrow n>\frac{1}{\varepsilon}$$

$$\varepsilon>0:\qquad n_0\in\mathbb{N}\wedge n_0>\frac{1}{\varepsilon}$$

$$0,11<0,12$$

$$\varepsilon=0,12\Rightarrow\ \frac{1}{\varepsilon}\approx 8,33\qquad n_0\geq 9\qquad \left|\frac{1}{n_0}-0\right|<0,12\ \Leftrightarrow\left|\frac{1}{9}-0\right|<0,12$$

$$\varepsilon=0,011\Rightarrow\ \frac{1}{\varepsilon}\approx 90,9\qquad n_0\geq 91\qquad \left|\frac{1}{n_0}-0\right|<0,011\Leftrightarrow\left|\frac{1}{91}-0\right|<0,011$$

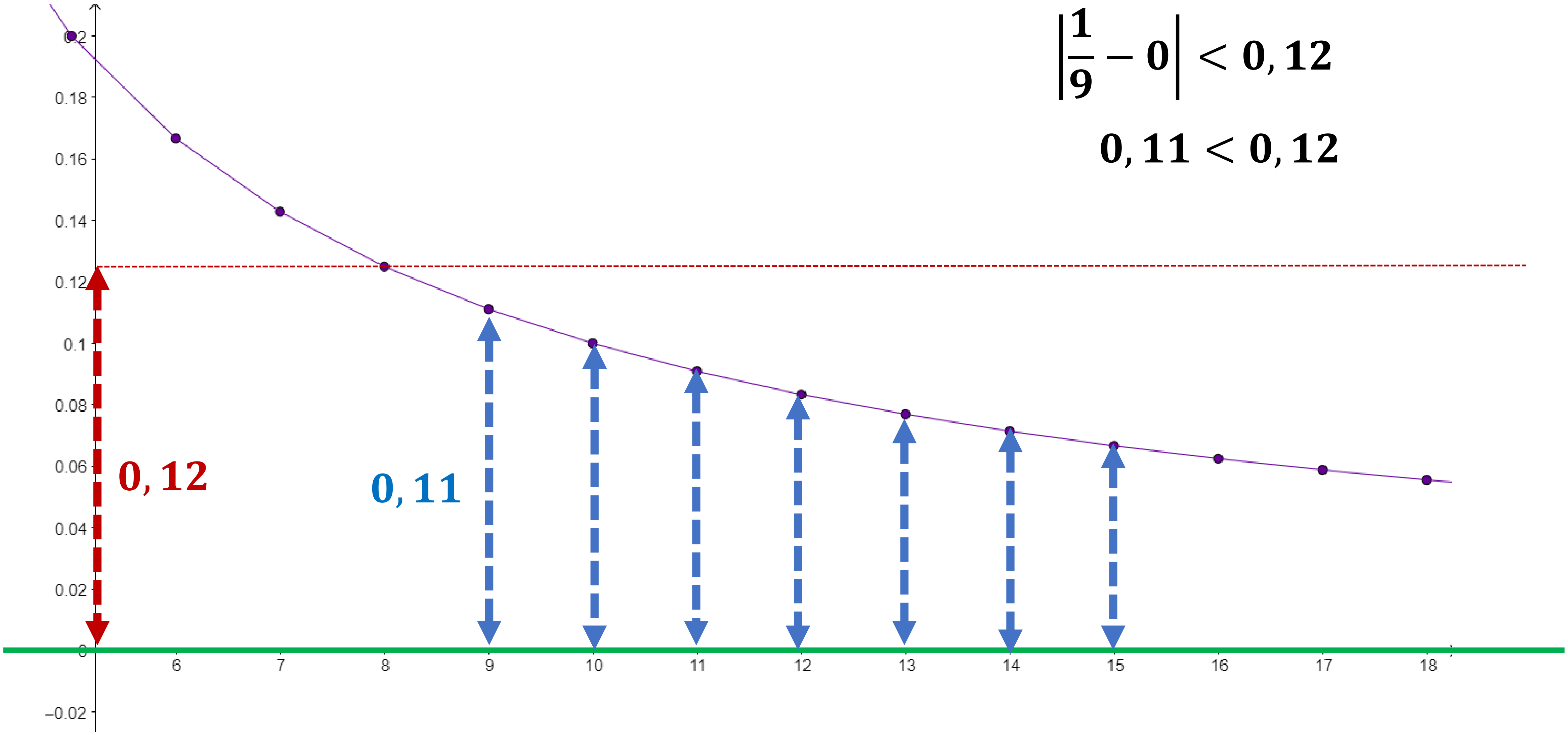
$$0,01091<0,011$$

$$\lim_{n\to\infty}\frac{1}{n}=0 \qquad \varepsilon=0,12 \qquad n_0=9$$

$$\left|\frac{1}{n_0}-0\right|<0,12$$

$$\left|\frac{1}{9}-0\right|<0,12$$

$$0,11<0,12$$



$$2 \times \infty = \infty \qquad (-3) \times \infty = -\infty \qquad \infty - 6 = \infty \qquad \infty \times \infty = \infty$$

Nedefinujeme: $\infty \times 0 \qquad \dfrac{\infty}{\infty} \qquad \infty^0 \qquad 0^0 \qquad \dfrac{0}{0} \qquad \infty - \infty \qquad 1^\infty$

Věta o aritmetice limit

(VOAL)

Jsou-li $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ **posloupnost,**

(i) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

(ii) $\lim\limits_{n\to\infty}(a_n . b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

Pokud pravá strana má smysl

(iii) $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$

Vypočtěte $\lim\limits_{n\to\infty} \frac{3n+5}{7n+11} \neq \dfrac{\lim\limits_{n\to\infty} 3n+5}{\lim\limits_{n\to\infty} 7n+11} = \frac{\infty}{\infty}$

$$\lim_{n\to\infty} \frac{3n+5}{7n+11} = \lim_{n\to\infty} \frac{\cancel{n}\left(3+\frac{5}{n}\right)}{\cancel{n}\left(7+\frac{11}{n}\right)} = \frac{\lim\limits_{n\to\infty}\left(3+\frac{5}{n}\right)}{\lim\limits_{n\to\infty}\left(7+\frac{11}{n}\right)} = \frac{\lim\limits_{n\to\infty} 3 + \lim\limits_{n\to\infty}\frac{5}{n}}{\lim\limits_{n\to\infty} 7 + \lim\limits_{n\to\infty}\frac{11}{n}}$$

$$= \frac{3+\lim\limits_{n\to\infty} 5 \cdot \lim\limits_{n\to\infty}\frac{1}{n}}{7+\lim\limits_{n\to\infty} 11 \cdot \lim\limits_{n\to\infty}\frac{1}{n}} = \frac{3+5.0}{7+11.0} = \frac{3}{7}$$

Vypočtěte $\lim\limits_{n\to\infty} \dfrac{\boldsymbol{n^2}}{\boldsymbol{n^2+1}} \neq \dfrac{\lim\limits_{n\to\infty} \boldsymbol{n^2}}{\lim\limits_{n\to\infty} \boldsymbol{n^2+1}} = \dfrac{\infty}{\infty}$

$$\lim_{n\to\infty} \frac{\boldsymbol{n^2}}{\boldsymbol{n^2+1}} = \lim_{n\to\infty} \frac{\cancel{\boldsymbol{n^2}}}{\cancel{\boldsymbol{n^2}}\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\boldsymbol{n^2}}\right)} = \lim_{n\to\infty} \frac{\boldsymbol{1}}{\boldsymbol{1}+\frac{\boldsymbol{1}}{\boldsymbol{n^2}}} = \frac{\lim\limits_{n\to\infty} \boldsymbol{1}}{\lim\limits_{n\to\infty}\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\boldsymbol{n^2}}\right)} = \frac{\boldsymbol{1}}{\lim\limits_{n\to\infty} \boldsymbol{1} + \lim\limits_{n\to\infty} \frac{\boldsymbol{1}}{\boldsymbol{n^2}}}$$

$$= \frac{\boldsymbol{1}}{\boldsymbol{1+0}} = 1$$