**Předmět:** NMTM101 Matematická analýza I

**Typ výuky:** Cvičení

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**Cvičení 3**

Znegujte následující výroky a rozhodněte, jestli platí výrok, nebo jeho negace:

***a***) $\forall x \in \mathbb{R}\, \forall y \in \mathbb{R}\colon\ x^2 + y^2 > 0$ **Výrok neplatí**

***Negace***:

$\neg\,(\forall x \in \mathbb{R}\, \forall y \in \mathbb{R}\colon\ x^2 + y^2 > 0)$

$\exists x \in \mathbb{R}\, \exists y \in \mathbb{R}\colon\ x^2 + y^2 \leq 0$ $\quad x = y = 0\colon\ \ x^2 + y^2 = 0 \leq 0$

**Negace platí**

$\boldsymbol{b)}\quad \forall \boldsymbol{x} \in \mathbb{R}\ \exists \boldsymbol{y} \in \mathbb{N}: (\boldsymbol{y} \leq \boldsymbol{x})\ \wedge (\boldsymbol{y} + \boldsymbol{1} > \boldsymbol{x})$

***Výrok neplatí***

***Negace:***

$\neg(\ \forall \boldsymbol{x} \in \mathbb{R}\ \exists \boldsymbol{y} \in \mathbb{N}: (\boldsymbol{y} \leq \boldsymbol{x})\ \wedge (\boldsymbol{y} + \boldsymbol{1} > \boldsymbol{x})\ )$

$\exists \boldsymbol{x} \in \mathbb{R}\ \ \forall \boldsymbol{y} \in \mathbb{N}: (\boldsymbol{y} > \boldsymbol{x})\ \vee (\boldsymbol{y} + \boldsymbol{1} \leq \boldsymbol{x})$

$\boldsymbol{x = 0}$: $\quad \boldsymbol{y > 0}\quad \vee\quad \boldsymbol{y \leq -1}$

***Negace platí***

**Je-li $M \subseteq \mathbb{R}$, pak:**

- $max M$ **je největší prvek množiny** $M$

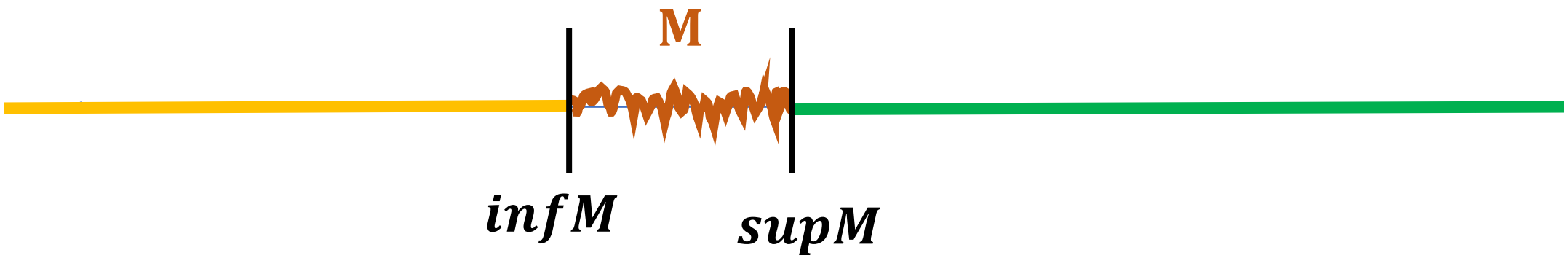
$$max M = a \iff a \in M \text{ a } \forall x \in M \; x \leq a$$

- $min M$ **je nejmenší prvek množiny** $M$

$$min M = b \iff b \in M \text{ a } \forall x \in M \; x \geq b$$

**Je-li $M \subseteq \mathbb{R}$, pak:**

- $supM$ **je nejmenší horní závora**
- $infM$ **je největší dolní závora**



**Nalezněte supremum, infimum, maximum a minimum následujících množin.**

$\boldsymbol{A = [2, \pi)}$ $\quad$ $\boldsymbol{B = \mathbb{N}}$ $\quad$ $\boldsymbol{C = \{3,\ 3.1,\ \sqrt{3},\ \ln(2),\ \pi,\ e,\ \frac{22}{7}\}}$

$\boldsymbol{D = (\pi, \frac{22}{7}] \cap \mathbb{Q}}$ $\quad$ $\boldsymbol{M = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}}$ $\quad$ $\boldsymbol{X = \{\frac{n}{n+1} : n \in \mathbb{N}\}}$

$\boldsymbol{Y = \{1 - \frac{n}{n+1} : n \in \mathbb{N}\}}$ $\quad$ $\boldsymbol{Z = \{x < \frac{1}{x} : x \in \mathbb{R}\}}$ $\quad$ $\boldsymbol{W = \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}}$

$A = [2, \pi)$




$inf(A) = 2$

$min(A) = 2$

$sup(A) = \pi$

$max(A)$ ✗

$$B = \mathbb{N}$$

$$B = \{1, 2, 3, 4, 5, \ldots\}$$

$$inf(B) = 1$$

$$min(B) = 1$$

$$sup(B) = \infty$$

$$max(B)$$ 

$$C = \{3,\quad 3.1,\quad \sqrt{3},\quad \ln(2),\quad \pi,\quad e,\quad \frac{22}{7}\}$$

$$C = \{3,\quad 3.1,\quad 1.732,\quad 0.693,\quad 3.1415,\quad 2.71,\quad 3.1428\}$$

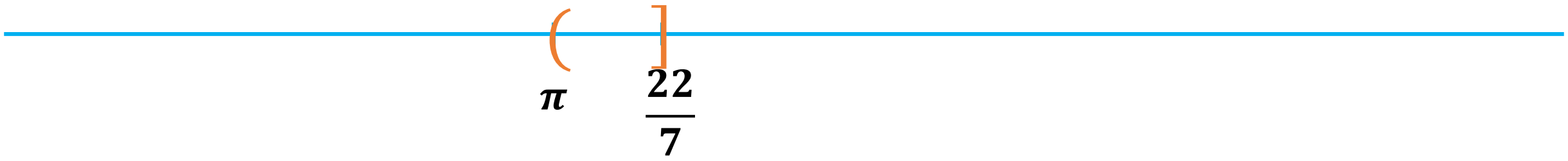
$$inf(C) = \ln(2)$$

$$min(C) = \ln(2)$$

$$sup(C) = \frac{22}{7}$$

$$max(C) = \frac{22}{7}$$

$$D = (\pi, \frac{22}{7}] \cap \mathbb{Q}$$




$inf(D) = \pi$

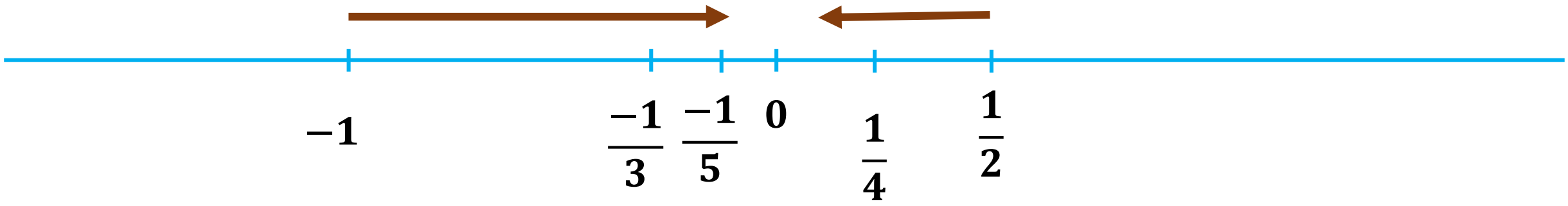
$min(D)$ ✗

$sup(D) = \frac{22}{7}$

$max(D) = \frac{22}{7}$

$$M = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$$

| $n \in \mathbb{N}$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\frac{(-1)^n}{n}$ | $-1$ | $\frac{1}{2}$ | $\frac{-1}{3}$ | $\frac{1}{4}$ | $\frac{-1}{5}$ |




$$inf(M) = -1$$

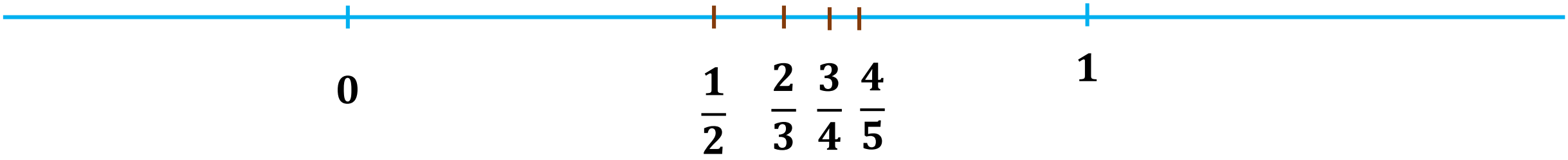
$$min(M) = -1$$

$$sup(M) = \frac{1}{2}$$

$$max(M) = \frac{1}{2}$$

$$X = \{\frac{n}{n+1} : n \in \mathbb{N}\}$$

| $n \in \mathbb{N}$ | 1 | 2 | 3 | 4 | 5 | $\longrightarrow$ | $n$ |
|---|---|---|---|---|---|---|---|
| $\frac{n}{n+1}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{3}{4}$ | $\frac{4}{5}$ | $\frac{5}{6}$ | $\longrightarrow$ | $\frac{n}{n+1} < 1$ |

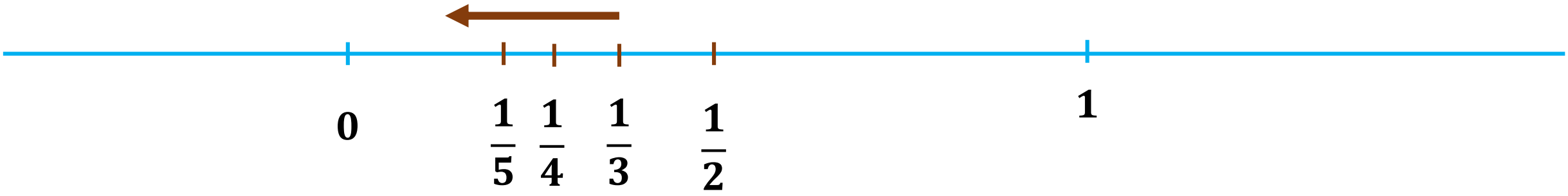



$$inf(X) = \frac{1}{2} \qquad sup(X) = 1$$

$$min(X) = \frac{1}{2} \qquad max(X) \quad \times$$

$$Y = \{1 - \frac{n}{n+1} : n \in \mathbb{N}\} = \{\frac{1}{n+1} : n \in \mathbb{N}\}$$

| $n \in \mathbb{N}$ | 1 | 2 | 3 | 4 | 5 | $\longrightarrow$ | $n$ |
|---|---|---|---|---|---|---|---|
| $\frac{1}{n+1}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\longrightarrow$ | $\frac{1}{n+1} > 0$ |




$inf(Y) = 0$

$sup(Y) = \frac{1}{2}$

$min(Y)$ ✗

$max(Y) = \frac{1}{2}$

$$Z = \{x < \frac{1}{x} : x \in \mathbb{R}\}$$

$$x^2 - 1 = 0 \Rightarrow x = 1 \vee x = -1$$

$$x < \frac{1}{x} \iff x - \frac{1}{x} < 0 \iff \frac{x^2 - 1}{x} < 0$$

$x = 0$

| | $(-\infty, -1)$ | $[-1, 0)$ | $[0, 1)$ | $[1, \infty)$ |
|---|---|---|---|---|
| $x^2 - 1$ | $+$ $0$ | $-$ | $-$ $0$ | $+$ |
| $x$ | $-$ | $-$ $0$ | $+$ | $+$ |
| $\frac{x^2 - 1}{x}$ | $-$ $0$ | $+$ $\infty$ | $-$ $0$ | $+$ |

$$Z = (-\infty, -1) \cup (0, 1)$$

$inf(Z) = -\infty$

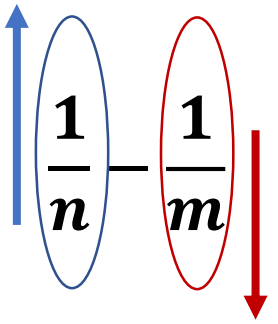
$sup(Z) = 1$

$min(Z)$ ✗

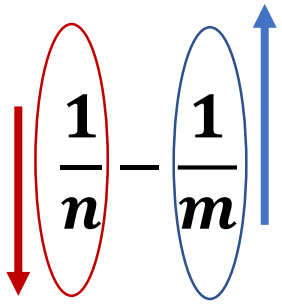
$max(Z)$ ✗

$$x \in (-\infty, -1) \cup (0, 1)$$

$$W = \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$$




$$n = 1: \quad \frac{1}{n} - \frac{1}{m} = 1 - \frac{1}{m} < 1$$




$$m = 1: \quad \frac{1}{n} - \frac{1}{m} = \frac{1}{n} - 1 > -1$$

$$-1 < \frac{1}{n} - \frac{1}{m} < 1$$

$sup(W) = 1$ $max(W)$ ✗

$inf(W) = -1$ $min(W)$ ✗

**Uvažujme libovolnou množinu $M$. Platí nutně následující nerovnost?**

$$\sup M \geq \inf M$$

$$M = \emptyset$$

$$\sup \emptyset = -\infty$$

$$\inf \emptyset = \infty$$



# LIMITA POSLOUPNOSTI

**Definice** (Vlastní limita posloupnosti)**.** Nechť $\{a_n\}_{n=1}^{\infty}$ je posloupnost. Řekneme, že číslo $A$ je limita posloupnosti $\{a_n\}_{n=1}^{\infty}$ a píšeme $\lim_{n\to\infty} a_n = A$ jestliže platí:

$$\forall \boldsymbol{\varepsilon > 0}\ \exists \boldsymbol{n_0} \in \mathbb{N}\ \forall \boldsymbol{n \geq n_0}\colon\ \boldsymbol{|a_n - A| < \varepsilon}$$

$$\lim_{n\to\infty}\frac{1}{n}=0$$

Podle definice limity posloupnosti

$$\forall\varepsilon>0\ \exists n_0\in\mathbb{N}\ \forall n\geq n_0:\ \left|\frac{1}{n}-0\right|<\varepsilon$$

$$\varepsilon>0:\qquad \left|\frac{1}{n}-0\right|<\varepsilon \quad\Leftrightarrow \frac{1}{n}<\varepsilon \quad\Leftrightarrow n>\frac{1}{\varepsilon}$$

$$\varepsilon>0:\qquad n_0\in\mathbb{N}\wedge n_0>\frac{1}{\varepsilon}$$

$$\varepsilon=0,12\Rightarrow\ \frac{1}{\varepsilon}\approx 8,33 \qquad n_0\geq 9 \qquad \left|\frac{1}{n_0}-0\right|<0,12\ \Leftrightarrow\left|\frac{1}{9}-0\right|<0,12$$

$$0,11<0,12$$

$$\varepsilon=0,011\Rightarrow\ \frac{1}{\varepsilon}\approx 90,9 \qquad n_0\geq 91 \qquad \left|\frac{1}{n_0}-0\right|<0,011\Leftrightarrow\left|\frac{1}{91}-0\right|<0,011$$

$$0,01091<0,011$$

$$\lim_{n\to\infty}\frac{1}{n}=0$$

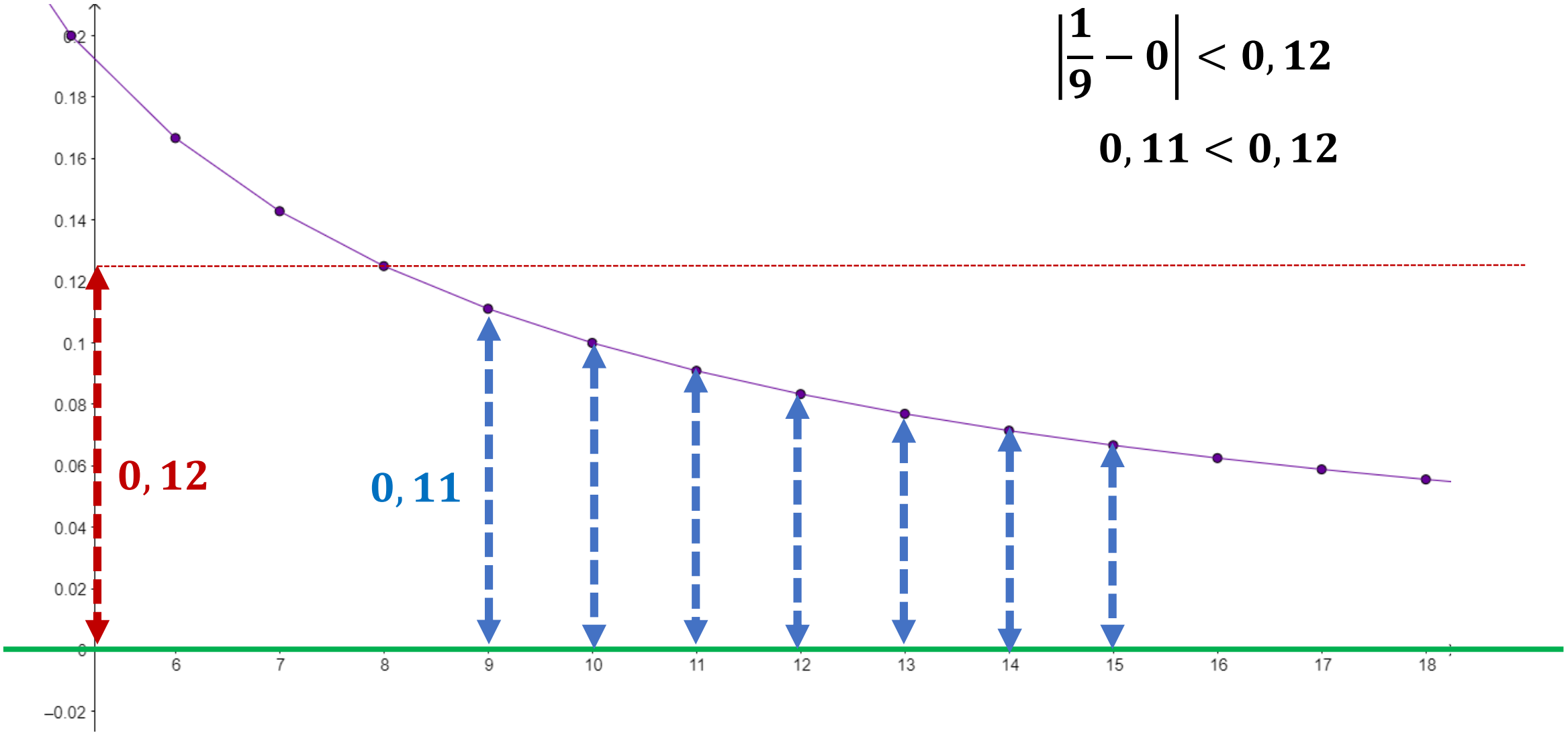
$$\varepsilon = 0,12$$

$$n_0 = 9$$

$$\left|\frac{1}{n_0}-0\right| < 0,12$$

$$\left|\frac{1}{9}-0\right| < 0,12$$

$$0,11 < 0,12$$



$$2 \times \infty = \infty \qquad (-3) \times \infty = -\infty \qquad \infty - 6 = \infty \qquad \infty \times \infty = \infty$$

**Nedefinujeme:** $\infty \times 0$ $\quad \frac{\infty}{\infty}$ $\quad \infty^0$ $\quad 0^0$ $\quad \frac{0}{0}$ $\quad \infty - \infty$ $\quad 1^\infty$

# Věta o aritmetice limit

# (VOAL)

**Jsou-li** $\{a_n\}_{n=1}^{\infty}$ , $\{b_n\}_{n=1}^{\infty}$ **posloupnost,**

**(i)** $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

**Pokud pravá strana má smysl**

**(ii)** $\lim_{n\to\infty}(a_n . b_n) = \lim_{n\to\infty} a_n \,.\, \lim_{n\to\infty} b_n$

**(iii)** $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

$$\lim_{n\to\infty}(a_n)^m = \left(\lim_{n\to\infty} a_n\right)^m$$

$$\lim_{n\to\infty}\sqrt[k]{a_n} = \sqrt[k]{\lim_{n\to\infty} a_n}$$

**Lemma** („Nulová · omezená = nulová.“).

Nechť $\{a_n\}_{n=1}^{\infty}$ a $\{b_n\}_{n=1}^{\infty}$ jsou posloupnosti

a nechť $\lim_{n\to\infty} a_n = 0$ a $\{b_n\}_{n=1}^{\infty}$ je omezená.

Potom $\lim_{n\to\infty} (a_n . b_n) = 0$

**Lemma** (O dvou policajtech).

Nechť $\{a_n\}_{n=1}^{\infty}$ , $\{b_n\}_{n=1}^{\infty}$ a $\{c_n\}_{n=1}^{\infty}$ jsou posloupnosti splňující následující podmínky:

(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a \in \mathbb{R}$;

(b) $\exists n_1 \in \mathbb{N}\ \forall n \geq n_1 : a_n \leq c_n \leq b_n$.

Pak je posloupnost $\{c_n\}_{n=1}^{\infty}$ konvergentní a jest $\lim_{n\to\infty} c_n = a$.

**Vypočtěte** $\lim_{n\to\infty} \frac{3n+5}{7n+11} \neq \frac{\lim_{n\to\infty} 3n+5}{\lim_{n\to\infty} 7n+11} = \frac{\infty}{\infty}$

$$\lim_{n\to\infty} \frac{3n+5}{7n+11} = \lim_{n\to\infty} \frac{\cancel{n}\,(3+\frac{5}{n})}{\cancel{n}\,(7+\frac{11}{n})} = \frac{\lim_{n\to\infty}(3+\frac{5}{n})}{\lim_{n\to\infty}(7+\frac{11}{n})} = \frac{\lim_{n\to\infty} 3 + \lim_{n\to\infty}\frac{5}{n}}{\lim_{n\to\infty} 7 + \lim_{n\to\infty}\frac{11}{n}}$$

$$= \frac{3 + \lim_{n\to\infty} 5 \cdot \lim_{n\to\infty}\frac{1}{n}}{7 + \lim_{n\to\infty} 11 \cdot \lim_{n\to\infty}\frac{1}{n}} = \frac{3+5.0}{7+11.0} = \frac{3}{7}$$

**Vypočtěte** $\lim\limits_{n\to\infty} \frac{n^2}{n^2+1} \neq \frac{\lim\limits_{n\to\infty} n^2}{\lim\limits_{n\to\infty} n^2+1} = \frac{\infty}{\infty}$

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}\left(1+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} \checkmark = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n^2}\right)} \checkmark = \frac{1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n^2}}$$

$$= \frac{1}{1+0} = 1$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3}=\lim_{n\to\infty}\frac{n^2\left(\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\left(\frac{3}{n}\right)^2+\cdots+1\right)}{\frac{n^3}{n}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\ \lim_{n\to\infty}\left(\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\left(\frac{3}{n}\right)^2+\cdots+1\right)$$

$$\neq 0\times\left[\lim_{n\to\infty}\left(\frac{1}{n}\right)^2+\lim_{n\to\infty}\left(\frac{2}{n}\right)^2+\cdots+\lim_{n\to\infty}1\right]$$

$$=0\times[0+0+\cdots+1]=0$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3}\neq\frac{\lim_{n\to\infty}(1^2+2^2+\cdots+n^2)}{\lim_{n\to\infty}n^3}=\frac{\infty}{\infty}$$

$$1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3}=\lim_{n\to\infty}\frac{\frac{n(n+1)(2n+1)}{6}}{n^3}=\lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3}$$

$$=\lim_{n\to\infty}\frac{n^3\left(2+\frac{3}{n}+\frac{1}{n^2}\right)}{6n^3}=\frac{\lim_{n\to\infty}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)}{\lim_{n\to\infty}6}=\cdots=\frac{2+0+0}{6}=\frac{1}{3}$$

$$\lim_{n\to\infty}\left(\frac{1+2+3+\cdots+n}{2+n}-\frac{n}{2}\right)\neq\frac{\lim_{n\to\infty}(1+2+3+\cdots+n)}{\lim_{n\to\infty}(2+n)}-\lim_{n\to\infty}\left(\frac{n}{2}\right)=\frac{\infty}{\infty}-\infty$$

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty}\left(\frac{1+2+3+\cdots+n}{2+n}-\frac{n}{2}\right)=\lim_{n\to\infty}\left(\frac{\frac{n(n+1)}{2}}{2+n}-\frac{n}{2}\right)=\lim_{n\to\infty}\left(\frac{n^2+n}{2(2+n)}-\frac{n}{2}\right)$$

$$=\lim_{n\to\infty}\frac{n^2+n-n(2+n)}{2(2+n)}=\lim_{n\to\infty}\frac{n^2+n-2n-n^2}{2(2+n)}=\lim_{n\to\infty}\frac{-n}{4+2n}=\lim_{n\to\infty}\frac{-n}{n\left(\frac{4}{n}+2\right)}$$

$$=\lim_{n\to\infty}\frac{-1}{\frac{4}{n}+2}=\frac{\lim_{n\to\infty}(-1)}{\lim_{n\to\infty}\left(\frac{4}{n}+2\right)}=\frac{-1}{\lim_{n\to\infty}4\,.\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}2}=\frac{-1}{4\,.\,0+2}=-\frac{1}{2}$$