Předmět: NMTM101 Matematická analýza I

Typ výuky: Cvičení

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Cvičení 11

$$\lim_{x\to 0}\left(\frac{1+tgx}{1+sinx}\right)^{\frac{1}{sin^3x}} = \cdots = e^{\frac{1}{2}}$$

$$\lim_{x\to \infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^{x} = \cdots = e$$

$$\lim_{x\to \pi}(-\cos x)^{cotg^2x} = \cdots = e^{\frac{-1}{2}}$$

$$\lim_{x\to 0}\left(\frac{1+tgx}{1+sinx}\right)^{\frac{1}{sin^3x}} \quad 1^{\infty}$$

$$\lim_{x\to 0}\frac{sinx}{x}=1 \qquad \lim_{x\to 1}\frac{\ln x}{x-1}=1$$

VOLSF(S) Exponenciální funkce e^y je spojitá v bodě $\frac{1}{2}$

$$=\lim_{x\to 0} exp\left(\frac{1}{sin^3x}\, ln\frac{1+tgx}{1+sinx}\right)=exp\left(\lim_{x\to 0}\frac{1}{sin^3x}\, ln\frac{1+tgx}{1+sinx}\right)=exp\left(\frac{1}{2}\right)=e^{\frac{1}{2}}$$

$$\frac{\frac{1+tgx}{1+sinx}-1}{x^3}=\frac{sinx(1-cosx)}{x^3(1+sinx)cosx}$$

$$\lim_{x\to 0}\frac{1}{sin^3x}\, ln\underbrace{\frac{1+tgx}{1+sinx}}_{\to 1}=\lim_{x\to 0}\frac{x^3}{sin^3x}\,\frac{ln\frac{1+tgx}{1+sinx}}{\frac{1+tgx}{1+sinx}-1}\,\frac{\frac{1+tgx}{1+sinx}-1}{x^3}$$

$$=\lim_{x\to 0}\left(\frac{x}{sinx}\right)^3\underbrace{\lim_{x\to 0}\frac{ln\frac{1+tgx}{1+sinx}}{\frac{1+tgx}{1+sinx}-1}}_{=1}\lim_{x\to 0}\frac{\frac{1+tgx}{1+sinx}-1}{x^3}=\cancel{\left(\lim_{x\to 0}\frac{sinx}{x}\right)^{-3}}^{\,1}.1.\lim_{x\to 0}\frac{sinx(1-cosx)}{x^3(1+sinx)cosx}$$

VOLSF(P)

$$x\to 0\Rightarrow y=\frac{1+tgx}{1+sinx}\to 1$$

$$=\lim_{x\to 0}\frac{sinx}{x}\lim_{x\to 0}\frac{1-cosx}{x^2}\lim_{x\to 0}\frac{1}{(1+sinx)cosx}$$

$$\lim_{y\to 1}\frac{lny}{y-1}=1 \qquad x\in P\left(0,\frac{\pi}{6}\right):\frac{1+tgx}{1+sinx}\neq 1$$

$$=1.\frac{1}{2}.1=\frac{1}{2}$$

Exponenciální funkce e^y je spojitá v bodě 1

VOLSF(S)

$$\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x = \lim_{x\to\infty} exp\left(x\, ln\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)\right) = exp\left(\lim_{x\to\infty} x\, ln\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)\right)$$

$$= exp(1) = e$$

$$\lim_{x\to\infty} x\, ln\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right) = \lim_{x\to\infty}\frac{ln\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{ln\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)}{\sin\frac{1}{x}+\cos\frac{1}{x}-1}\cdot\frac{\sin\frac{1}{x}+\cos\frac{1}{x}-1}{\frac{1}{x}}$$

VOLSF(P) $= 1$ VOLSF(P) $= 1$

$$= \lim_{x\to\infty}\frac{ln\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)}{\sin\frac{1}{x}+\cos\frac{1}{x}-1}\left(\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}+\lim_{x\to\infty}\frac{\cos\frac{1}{x}-1}{\frac{1}{x}}\right) = 1.\left(1-\lim_{x\to\infty}\frac{1-\cos\frac{1}{x}}{\left(\frac{1}{x}\right)^2}.\left(\frac{1}{x}\right)\right) =$$

$$x\to\infty \Rightarrow y=\frac{1}{x}\to 0$$

$$\lim_{y\to 0}\frac{siny}{y}=1 \qquad \frac{1}{x}\neq 0$$

$$x\to\infty \Rightarrow y=\sin\frac{1}{x}+\cos\frac{1}{x}\to 1$$

$$\lim_{y\to 1}\frac{lny}{y-1}=1$$

$$\forall x\in(\pi,\infty): \sin\frac{1}{x}+\cos\frac{1}{x}\neq 1$$

VOLSF(P) $= \frac{1}{2}$ $= 0$

$$1-\lim_{x\to\infty}\frac{1-\cos\frac{1}{x}}{\left(\frac{1}{x}\right)^2}\lim_{x\to\infty}\left(\frac{1}{x}\right) = 1$$

VOLSF(S) Exponenciální funkce e^y je spojitá v bodě $\frac{-1}{2}$

$$\lim_{x\to\pi}(-\cos x)^{cotg^2 x} = \lim_{x\to\pi} exp(cotg^2 x \ \ln(-\cos x)) = exp(\lim_{x\to\pi} cotg^2 x \ln(-\cos x))$$

$$= exp\left(\frac{-1}{2}\right) = e^{\frac{-1}{2}}$$

$$\lim_{x\to\pi} cotg^2 x \ln(-\cos x)) = \lim_{x\to\pi} \frac{cos^2(x)}{sin^2(x)} ln(-cosx)$$

$\boxed{x - \pi = y}$ $\boxed{x = \pi + y}$

$\boxed{x \to \pi \Longrightarrow y \to 0}$

$$= \lim_{y\to 0} \frac{cos^2(\pi + y)}{sin^2(\pi + y)} ln(-cos(\pi + y)) =$$

$\boxed{cos^2 \text{ a } sin^2 \text{ jsou } \pi\text{-periodické fce.}}$

$\boxed{cos^2(\pi + y) = cos^2(y)}$

$\boxed{sin^2(\pi + y) = sin^2(y)}$

$cos(\pi + y) = -cosy$

$$= \lim_{y\to 0} \frac{cos^2(y)}{sin^2(y)} ln(cosy) = \lim_{y\to 0} cos^2 y \lim_{y\to 0} \frac{y^2}{sin^2 y} \frac{ln(cos\, y)}{cosy - 1} \frac{cosy - 1}{y^2}$$

$$= 1.\left(\lim_{y\to 0} \frac{siny}{y}\right)^{-2} . \lim_{y\to 0} \frac{ln(cos\, y)}{cosy - 1} . \left(-\lim_{y\to 0} \frac{1 - cosy}{y^2}\right) = \frac{-1}{2}$$

Jednostranné limity funkcí

$$\lim_{x\to 0^+}\frac{1}{1+2^{\frac{1}{x}}}=\frac{\lim\limits_{x\to 0^+}1}{\lim\limits_{x\to 0^+}1+\lim\limits_{x\to 0^+}2^{\frac{1}{x}}}=\frac{1}{1+2^{+\infty}}=\frac{1}{1+\infty}=0$$

$$\lim_{x\to 0^-}\frac{1}{1+2^{\frac{1}{x}}}=\frac{\lim\limits_{x\to 0^-}1}{\lim\limits_{x\to 0^-}1+\lim\limits_{x\to 0^-}2^{\frac{1}{x}}}=\frac{1}{1+2^{-\infty}}=\frac{1}{1+0}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{1}{1+2^{\frac{1}{x}}}$$

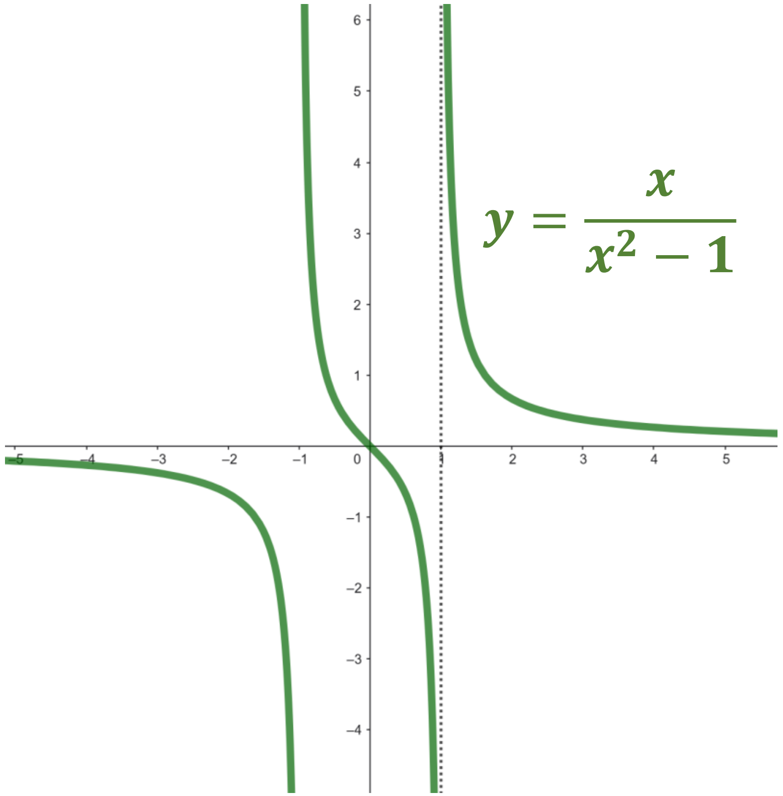
Neexsistuje

$$\lim_{x\to 1^+}\frac{x}{x^2-1}=\frac{1}{0^+}=+\infty$$

$$\lim_{x\to 1}\frac{x}{x^2-1}$$

Neexsistuje

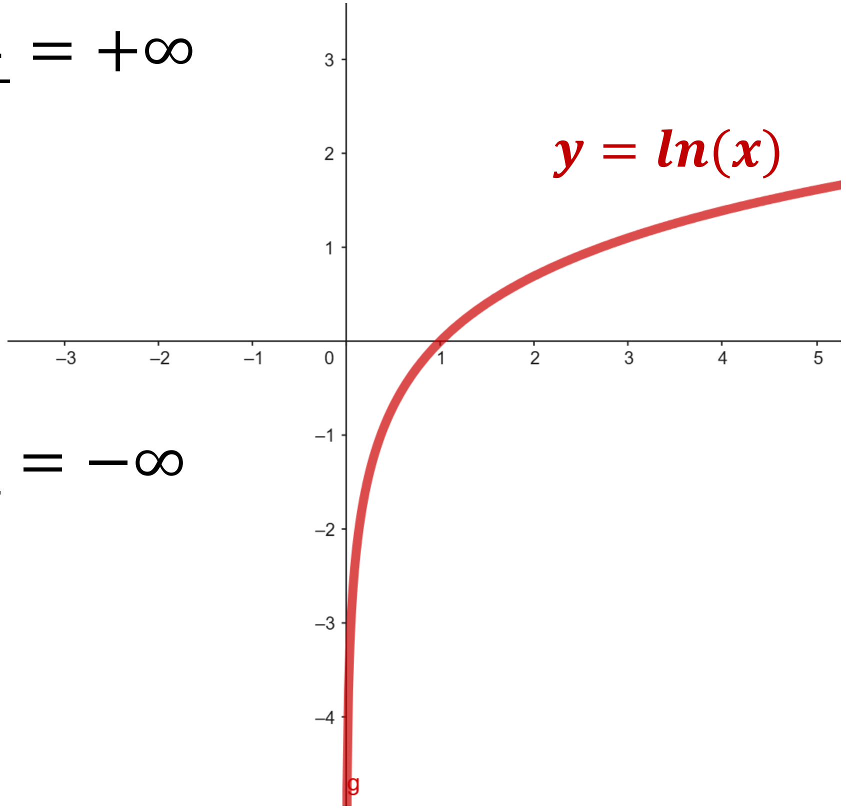
$$\lim_{x\to 1^-}\frac{x}{x^2-1}=\frac{1}{0^-}=-\infty$$



$$\lim_{x\to 2^+}\frac{x^2+x-6}{|x-2|}=\lim_{x\to 2^+}\frac{\cancel{(x-2)}(x+3)}{\cancel{x-2}}=5$$

$$x\to 2^+ \Rightarrow\ x-2\to 0^+\ \ ,x-2>0\Rightarrow |x-2|=x-2$$

$$\lim_{x\to 2^-}\frac{x^2}{ln(3-x)}=\frac{4}{ln(1^+)}=\frac{4}{0^+}=+\infty$$



$$x\to 2^- \Rightarrow\ 3-x\to 1^+$$

$$\lim_{x\to 2^+}\frac{x^2}{ln(3-x)}=\frac{4}{ln(1^-)}=\frac{4}{0^-}=-\infty$$

$$x\to 2^+ \Rightarrow\ 3-x\to 1^-$$

Limity posloupností (Heineho věta):

$$\lim_{n\to\infty} n(\sqrt[n]{2}-1) = \lim_{n\to\infty} n(2^{\frac{1}{n}}-1) \overset{\text{H.V}}{=} \lim_{x\to\infty} x(2^{\frac{1}{x}}-1)$$

$$= \lim_{x\to\infty}\frac{2^{\frac{1}{x}}-1}{\frac{1}{x}} = \lim_{x\to\infty}\overbrace{\frac{e^{\frac{1}{x}ln2}-1}{\frac{1}{x}ln2}}^{\to 1}\, ln2 = ln2$$

VOLSF(P)

Řady

$$\sum_{n=1}^{\infty} a_n$$

Geometrická řada

$$\sum_{n=0}^{\infty} q^n = \begin{cases} \dfrac{1}{1-q} & -1 < q < 1 \\ \text{Divergentní} & q \geq 1 \lor q \leq -1 \end{cases}$$

$$a + aq + aq^2 + \ldots + aq^n + \cdots = \frac{a}{1-q}$$

$$-1 < q < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \quad \begin{cases} \textbf{Konvergentní} & \alpha > 1 \\ \textbf{Divergentní} & \alpha \leq 1 \end{cases}$$

$$\sum_{n=1}^{\infty} \frac{1}{n\,(\log n)^{a}} \quad \begin{cases} \textbf{Konvergentní} & a > 1 \\ \textbf{Divergentní} & a \leq 1 \end{cases}$$

Věta (Nutná podmínka konvergence řady).

Nechť $\sum_{\boldsymbol{n=1}}^{\infty} \boldsymbol{a_n}$ konverguje, pak $\lim\limits_{n\to\infty} a_n = 0$

$$\boldsymbol{\sum_{n=1}^{\infty} a_n \textbf{ konverguje} \Rightarrow \lim_{n\to\infty} a_n = 0}$$

$$\boldsymbol{\lim_{n\to\infty} a_n \neq 0 \quad \textbf{nebo} \lim_{n\to\infty} \mathrm{a_n} \textbf{ neexistuje} \quad \Rightarrow \sum_{n=1}^{\infty} a_n \textbf{ diverguje}}$$

$$\boldsymbol{\lim_{n\to\infty} a_n = 0 \quad \Rightarrow \quad \textbf{Nemůžeme nic říct (možná konverguje nebo diverguje)}}$$

$$\sum_{n=1}^{\infty}\frac{n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n} \quad \text{diverguje}$$

harmonická řada

$$\lim_{n\to\infty} a_n = 0 \quad \text{a řada diverguje}$$

$$\sum_{n=1}^{\infty}\frac{n+1}{2n+4}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n+1}{2n+4} = \frac{1}{2} \neq 0 \qquad \sum_{n=1}^{\infty}\frac{n+1}{2n+4} \quad \text{diverguje}$$

$$\sum_{n=1}^{\infty} 2^{-n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$\lim_{n\to\infty} a_n = 0 \quad \text{a řada konverguje}$$

geometrická řada

$$\frac{1}{2} < 1$$

$$\sum_{n=1}^{\infty}(1+\frac{1}{n})^n \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty}(1+\frac{1}{n})^n = e \neq 0 \qquad \sum_{n=1}^{\infty}(1+\frac{1}{n})^n$$ **diverguje**

$$\sum_{n=1}^{\infty}(\frac{5}{6})^{\frac{n}{2}} = \sum_{n=1}^{\infty}(\sqrt{\frac{5}{6}})^n = \frac{\sqrt{\frac{5}{6}}}{1-\sqrt{\frac{5}{6}}}$$ **geometrická řada** $\sqrt{\frac{5}{6}} < 1$

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}} \qquad \frac{3}{2} > 1$$ **Konvergentní**

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}} \qquad \frac{1}{2} < 1$$ **Divergentní**

$$\sum_{n=1}^{\infty} \frac{1}{n\, log n} \quad \textbf{Divergentní}$$

$$\sum_{n=1}^{\infty} \frac{1}{n\,(log n)^{a}} \quad \begin{cases} \textbf{Konvergentní} & a > 1 \\ \textbf{Divergentní} & a \leq 1 \end{cases}$$

$$\sum_{n=1}^{\infty}(1+(\frac{1}{6})^n) \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty}(1+(\frac{1}{6})^n) = 1 \neq 0$$ **Divergentní**

$$\sum_{n=1}^{\infty}\frac{3n}{\sqrt{n}} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3n}{\sqrt{n}} = 3\lim_{n\to\infty}\sqrt{n} = \infty \neq 0$$ **Divergentní**

$$\sum_{n=1}^{\infty}\frac{1}{n\,log^2 n}$$ **Konvergentní**

$$2 > 1$$