Předmět: NMTM101 Matematická analýza I

Typ výuky: Cvičení

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Cvičení 10

Věta o limitě složené funkce (VOLSF)

VOLSF(S) VOLSF(P)

Známé limity:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{\ln(x + 1)}{x} = 1 \qquad \lim_{x \to 1} \frac{\ln x}{x - 1} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0}\frac{5^x-1}{x} \qquad \frac{0}{0}$$

$$a^x = e^{\ln a^x} = e^{x\ln a}$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

$$=\lim_{x\to 0}\frac{(e^{x\ln 5}-1)\ln 5}{x\ln 5} = \ln 5 \lim_{x\to 0}\frac{e^{x\ln 5}-1}{x\ln 5} = \ln 5\times 1 = \ln 5$$

VOLSF(P)

$$\boxed{y = x\ln 5 \to 0}$$

Existuje okolí $U_\varepsilon^*(0) \Rightarrow x\ln 5 \neq 0$

$$\boxed{\lim_{y\to 0}\frac{e^y-1}{y}=1}$$

$$\lim_{x\to 0}\frac{\ln(x^2+e^{2x^2})}{\ln(x^2+e^{3x^2})} \qquad \frac{ln(1)}{ln(1)}=\frac{0}{0}$$

$$\boxed{\lim_{x\to 1}\frac{\ln x}{x-1}=1} \qquad \boxed{\lim_{x\to 0}\frac{e^x-1}{x}=1}$$

$$=\lim_{x\to 0}\frac{\dfrac{\ln(x^2+e^{2x^2})}{x^2+e^{2x^2}-1}(x^2+e^{2x^2}-1)}{\dfrac{\ln(x^2+e^{3x^2})}{x^2+e^{3x^2}-1}(x^2+e^{3x^2}-1)}=\frac{\overbrace{\lim_{x\to 0}\dfrac{\ln(x^2+e^{2x^2})}{x^2+e^{2x^2}-1}}^{=1}}{\underbrace{\lim_{x\to 0}\dfrac{\ln(x^2+e^{3x^2})}{x^2+e^{3x^2}-1}}_{=1}}\lim_{x\to 0}\frac{\cancel{x^2}\left(1+\dfrac{e^{2x^2}-1}{x^2}\right)}{\cancel{x^2}\left(1+\dfrac{e^{3x^2}-1}{x^2}\right)}$$

VOLSF(P)

VOLSF(P)

$$=\frac{1+2\overbrace{\lim_{x\to 0}\dfrac{e^{2x^2}-1}{2x^2}}^{=1}}{1+3\underbrace{\lim_{x\to 0}\dfrac{e^{3x^2}-1}{3x^2}}_{=1}}=\frac{1+2}{1+3}=\frac{3}{4}$$

VOLSF(P)

VOLSF(P)

$$\boxed{y=x^2+e^{3x^2}\to 1}$$

$$\boxed{\lim_{y\to 1}\frac{lny}{y-1}=1}$$

$$x^2+e^{3x^2}\geq e^{3x^2}>e^0=1$$

$$x\neq 0$$

$$\lim_{x\to 1}\frac{\ln(x)}{x-1}=1 \qquad \lim_{x\to 1}\frac{1-\cos x}{x^2}=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{\ln(\cos x)}{3x^2} \qquad \frac{0}{0}$$

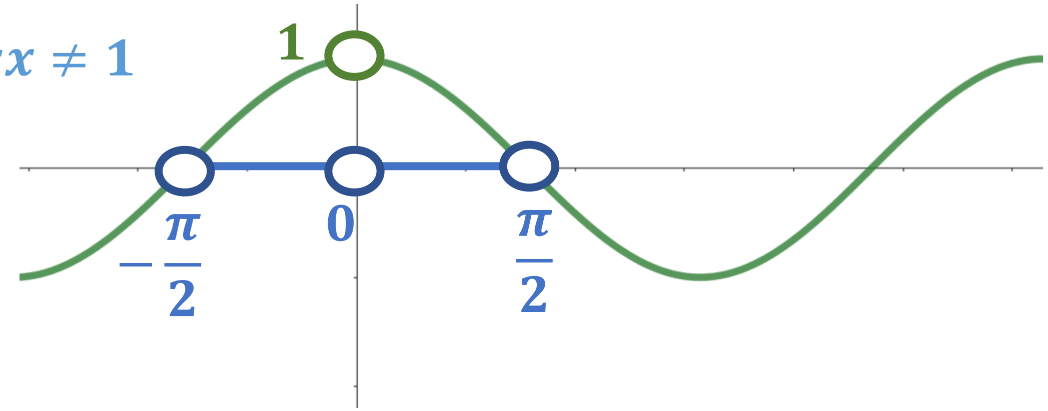
$$=\frac{1}{3}\lim_{x\to 0}\frac{\ln(\cos x)\,(\cos x-1)}{(\cos x-1)\,x^2}=\frac{1}{3}\underbrace{\lim_{x\to 0}\frac{\ln(\cos x)}{\cos x-1}}_{=1}\lim_{x\to 0}\frac{\cos x-1}{x^2}=\frac{1}{3}(1)\left(-\frac{1}{2}\right)=-\frac{1}{6}$$

VOLSF(P)

$$\boxed{y=\cos x\to 1}$$

$$\boxed{\lim_{y\to 1}\frac{\ln(y)}{y-1}=1}$$

$$x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)-\{0\}:\cos x\neq 1$$



$$\lim_{x\to 0}\frac{e^{x^2 \sin\frac{1}{x}}-1}{x^2 \sin\frac{1}{x}}$$

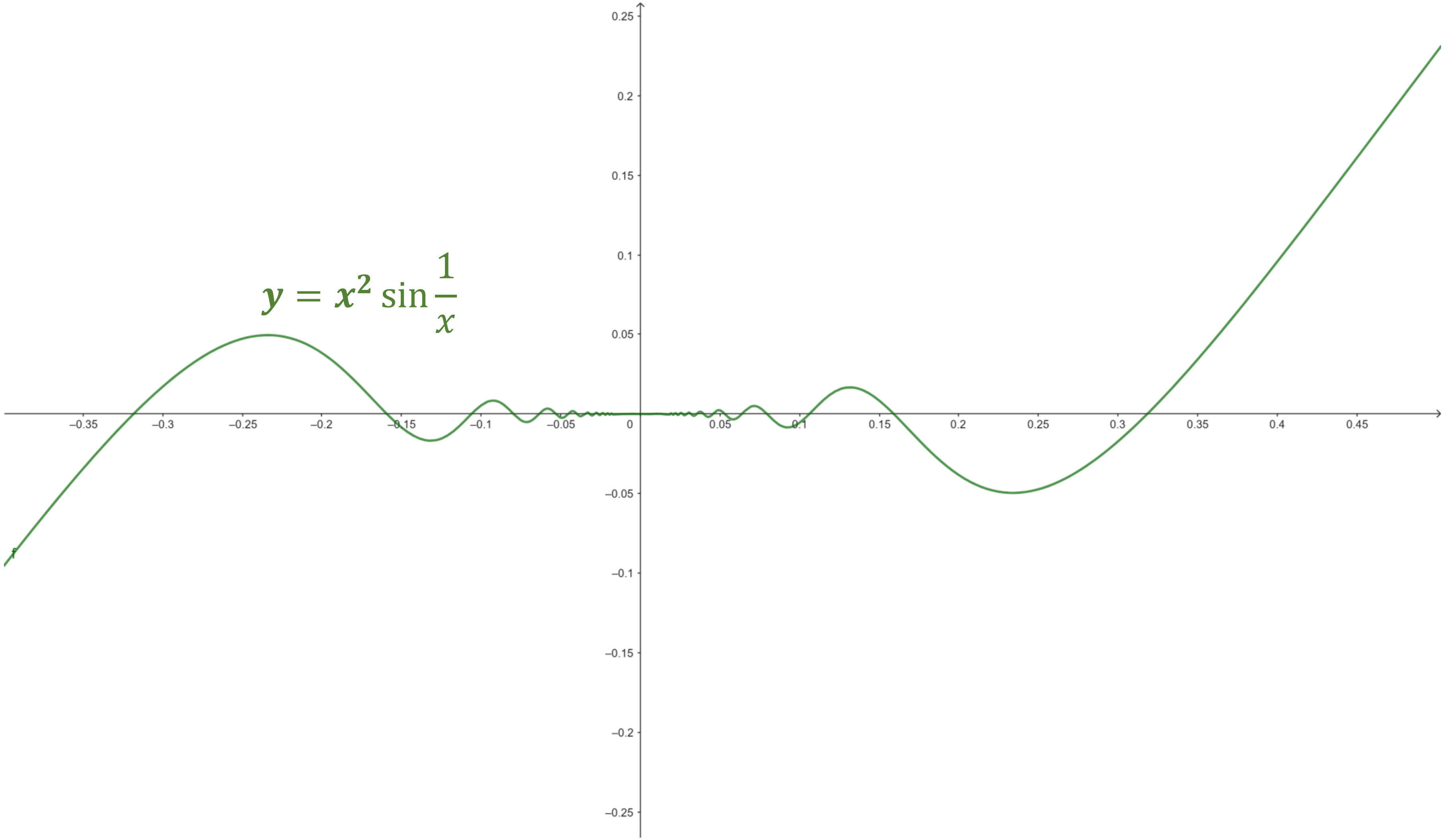
Neexsituje

$$\forall P(0,\delta), \exists x \in P(0,\delta): x^2 \sin\frac{1}{x} = 0$$

$$x = \frac{1}{n\pi}: x^2 \sin\frac{1}{x} = \left(\frac{1}{n\pi}\right)^2 \sin n\pi = 0$$

$$\boxed{y = x^2 \sin\frac{1}{x} \to 0}$$

$$\boxed{\lim_{y\to 0}\frac{e^y-1}{y} = 1}$$



Limity typu „1^{∞}“:

$$\underbrace{f(x)}_{\to 1}{}^{\overbrace{g(x)}^{\to \infty}} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)} = exp(g(x)\ln f(x))$$

$$\lim_{x\to\infty}(1+\frac{1}{x})^{x} = e$$

Exponenciální funkce e^{y}
je spojitá v bodě 1

VOLSF(S)

$$\lim_{x\to\infty}(1+\frac{1}{x})^{x} = \lim_{x\to\infty} exp(x\ln(1+\frac{1}{x})) = exp(\lim_{x\to\infty} x\ln(1+\frac{1}{x})) = exp(1) = e$$

$$\lim_{x\to\infty} x\ln(1+\frac{1}{x}) = \lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = 1$$

VOLSF(P)

$$\boxed{y = \frac{1}{x} \to 0} \quad \boxed{\lim_{y\to 0}\frac{ln(1+y)}{y} = 1}$$

$$\forall P(\infty,\delta):\ \frac{1}{x} \neq 0$$

$$\lim_{x\to0}\frac{\ln(1+x)}{x}=1$$

$$\lim_{x\to\infty}\left(1+\frac{1}{3x-1}\right)^{2x} \quad 1^{\infty}$$

Exponenciální funkce e^{y} je spojitá v bodě $\frac{2}{3}$

VOLSF(S)

$$=\lim_{x\to\infty} exp\left(2x\,ln\left(1+\frac{1}{3x-1}\right)\right)=exp\left(\lim_{x\to\infty}2x ln\left(1+\frac{1}{3x-1}\right)\right)=exp\left(\frac{2}{3}\right)=e^{\frac{2}{3}}$$

$$\lim_{x\to\infty}2x ln\left(1+\frac{1}{3x-1}\right)=$$

VOLSF(P)

$$=\lim_{x\to\infty}\frac{2x\,ln\left(1+\frac{1}{3x-1}\right)\left(\frac{1}{3x-1}\right)}{\frac{1}{3x-1}}\text{VOAL}=\lim_{x\to\infty}\frac{ln\left(1+\frac{1}{3x-1}\right)}{\frac{1}{3x-1}}\lim_{x\to\infty}\frac{2x}{3x-1}=1\cdot\frac{2}{3}=\frac{2}{3}$$

$$y=\frac{1}{3x-1}\to0 \qquad \frac{1}{3x-1}\neq0$$

$$\lim_{y\to0}\frac{ln(1+y)}{y}=1$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = ? \quad 1^\infty$$

$a \neq 0$

Exponenciální funkce e^y
je spojitá v bodě a

VOLSF(S)

$$= \lim_{x\to\infty} exp\left(x\, ln\left(1+\frac{a}{x}\right)\right) = exp\left(\lim_{x\to\infty} x\, ln\left(1+\frac{a}{x}\right)\right) = exp(a) = e^a$$

$$\lim_{x\to\infty} x ln\left(1+\frac{a}{x}\right) =$$

VOLSF(P)

$$= \lim_{x\to\infty}\frac{x\, ln\left(1+\frac{a}{x}\right)\frac{a}{x}}{\frac{a}{x}} = \lim_{x\to\infty}\frac{ln\left(1+\frac{a}{x}\right)}{\frac{a}{x}}\lim_{x\to\infty}\frac{ax}{x} = 1\,.\,a = a$$

$y = \frac{a}{x} \to 0 \qquad \frac{a}{x} \neq 0$

$$\lim_{y\to 0}\frac{ln(1+y)}{y} = 1$$

$$\lim_{x\to 0}\left(\frac{3^x+5^x}{2}\right)^{\frac{1}{x}} = \cdots = \sqrt{15}$$

$$\lim_{x\to 0}\left(\frac{1+tgx}{1+sinx}\right)^{\frac{1}{sin^3x}} = \cdots = e^{\frac{1}{2}}$$

$$\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^{x} = \cdots = e$$

$$\lim_{x\to\pi}(-\cos x)^{cotg^2x} = \cdots = e^{\frac{-1}{2}}$$

VOLSF(S)

$$\lim_{x\to 0}\left(\frac{3^x+5^x}{2}\right)^{\frac{1}{x}} = \lim_{x\to 0} exp\left(\frac{1}{x}\ln\left(\frac{3^x+5^x}{2}\right)\right) = exp\left(\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{3^x+5^x}{2}\right)\right) = exp(ln\sqrt{15}) = \sqrt{15}$$

Exponenciální funkce e^y
je spojitá v bodě $ln\sqrt{15}$

$$\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{3^x+5^x}{2}\right) =$$

VOLSF(P)

$$\lim_{x\to 0}\frac{\frac{3^x+5^x}{2}-1}{x}\cdot\frac{\ln\frac{3^x+5^x}{2}}{\frac{3^x+5^x}{2}-1} = \lim_{x\to 0}\frac{3^x+5^x-2}{2x}\left(\lim_{x\to 0}\frac{\ln\frac{3^x+5^x}{2}}{\frac{3^x+5^x}{2}-1}\right)$$

$= 1$

$$\boxed{x\to 0 \Rightarrow y = \frac{3^x+5^x}{2}\to 1}$$

$$\boxed{\lim_{y\to 1}\frac{lny}{y-1} = 1}$$

$$\forall P(0,\delta): \frac{3^x+5^x}{2}\neq 1$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{3^x-1+5^x-1}{x} = \frac{1}{2}\lim_{x\to 0}\left(\frac{3^x-1}{x}+\frac{5^x-1}{x}\right)$$

$$= \frac{1}{2}\left(\lim_{x\to 0}\frac{e^{x\,ln\,3}-1}{ln\,3\,.x}ln\,3 + \lim_{x\to 0}\frac{e^{x\,ln\,5}-1}{ln\,5\,.x}ln\,5\right) = \frac{1}{2}(\ln 3+\ln 5) = \frac{1}{2}\ln 15 = ln\sqrt{15}$$

VOLSF(P) $x\,ln\,3\neq 0$

VOLSF(P) $x\,ln\,5\neq 0$