

Předmět: NMTM102 Matematická analýza II

Typ výuky: Cvičení

Vahid Borji

borji@karlin.mff.cuni.cz

Cvičení 8 (8.4.2026)

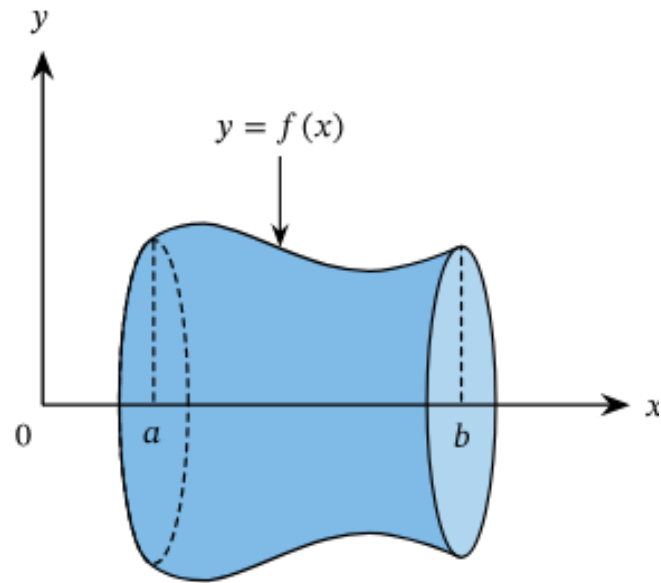
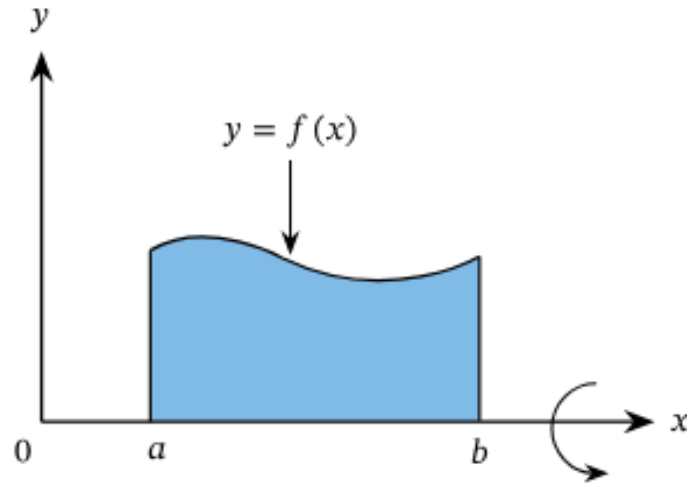
Nějaké aplikace:

- Objem rotačního tělesa $(V = \int_a^b \pi(f(x))^2 dx)$

- Povrch rotačního tělesa $(S = 2\pi \cdot \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2} dx)$

Objem rotačního tělesa

$$V = \int_a^b \pi(f(x))^2 dx$$



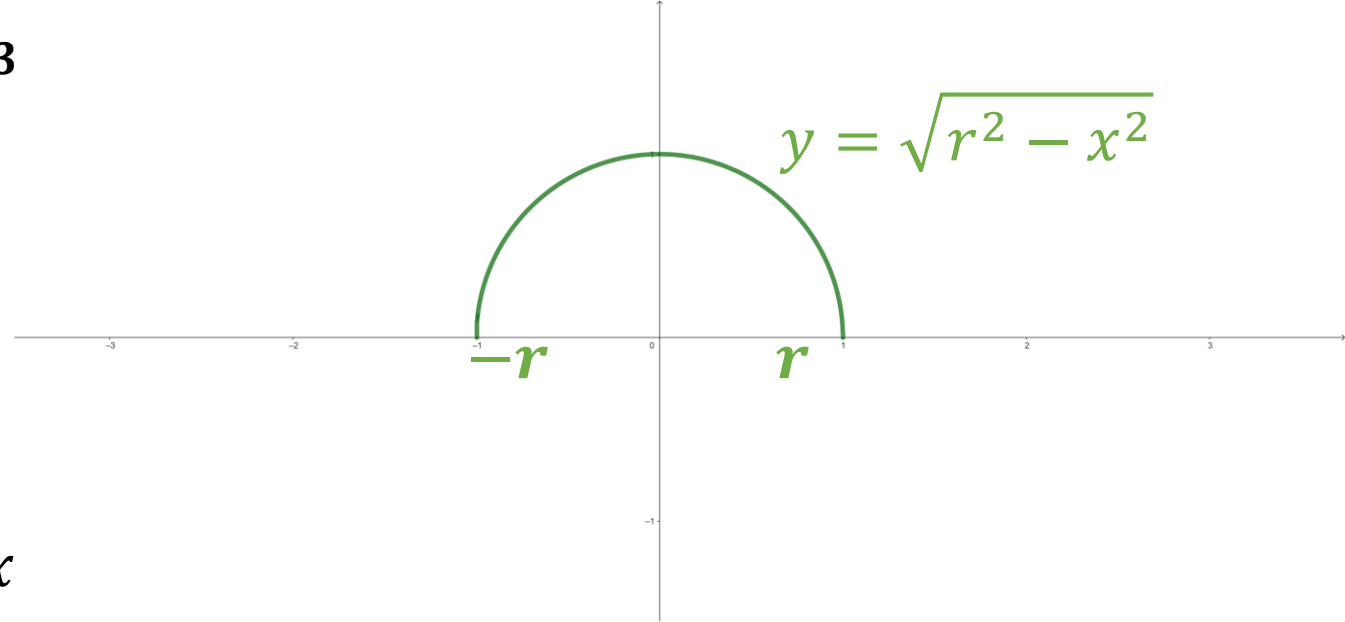
Ukažte, že objem koule o poloměru r je $\frac{4}{3}\pi r^3$

$$V = \int_{-r}^r \pi(\sqrt{r^2 - x^2})^2 dx$$

$$= \int_{-r}^r \pi(r^2 - x^2) dx = 2\pi \int_0^r (r^2 - x^2) dx$$

$$= 2\pi \left(r^2 x - \frac{x^3}{3} \right) \Big|_0^r = 2\pi \left(r^3 - \frac{r^3}{3} - 0 \right)$$

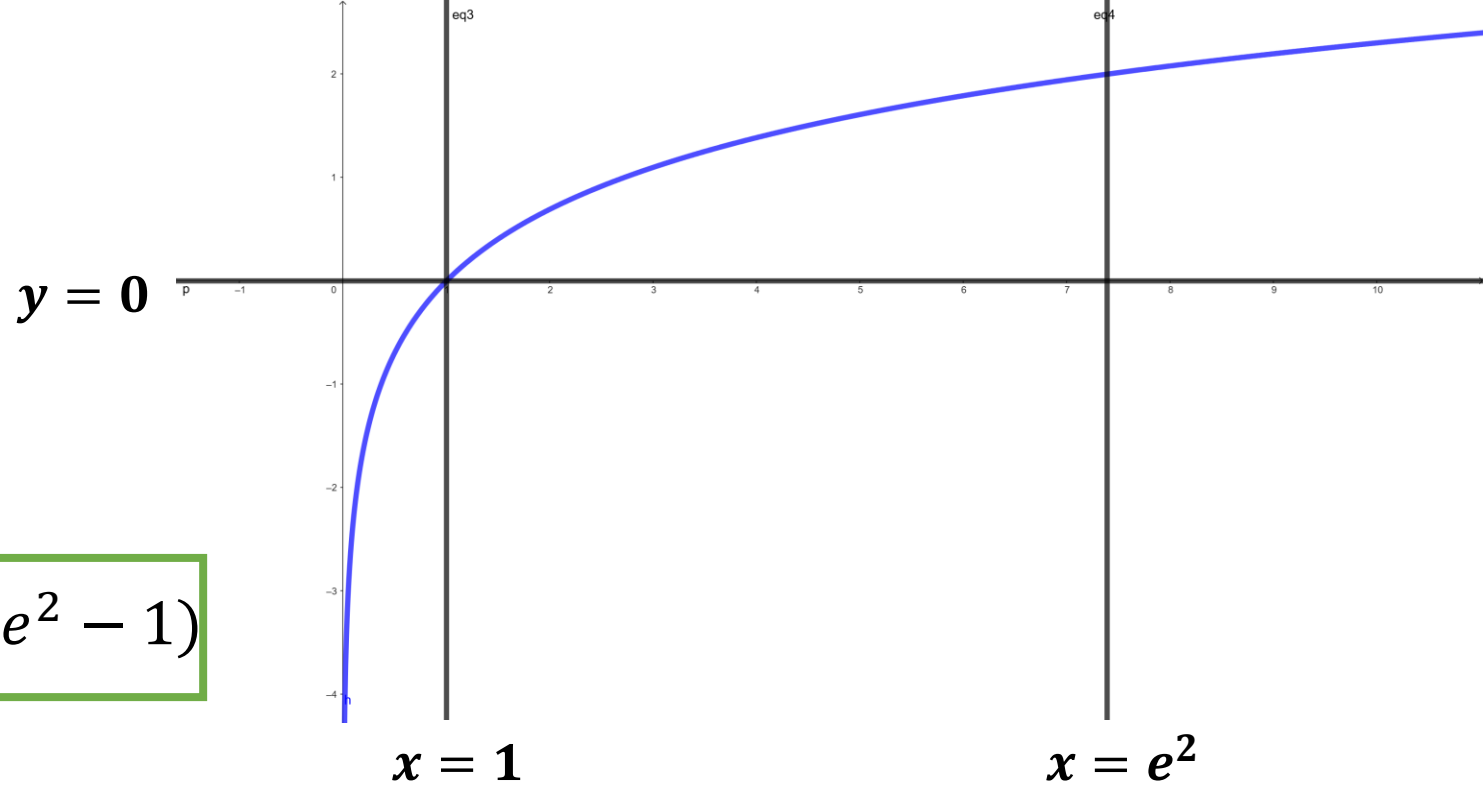
$$= 2\pi \frac{2r^3}{3} = \frac{4}{3}\pi r^3$$



Spočtete objem tělesa vzniklého rotací kolem osy x plochy určené křivkou $y = \ln x$, $y = 0$, $x \in [1, e^2]$.

$$V = \int_1^{e^2} \pi (\ln x)^2 dx = \pi \int_1^{e^2} (\ln x)^2 dx$$

$$= \pi (x(\ln x)^2 - 2x \ln x + 2x) \Big|_1^{e^2} = 2\pi(e^2 - 1)$$



$$\int (\ln x)^2 dx = \int 1 \cdot (\ln x)^2 dx = x(\ln x)^2 - 2 \int \ln x dx \stackrel{C}{=} x(\ln x)^2 - 2(x \ln x - x)$$

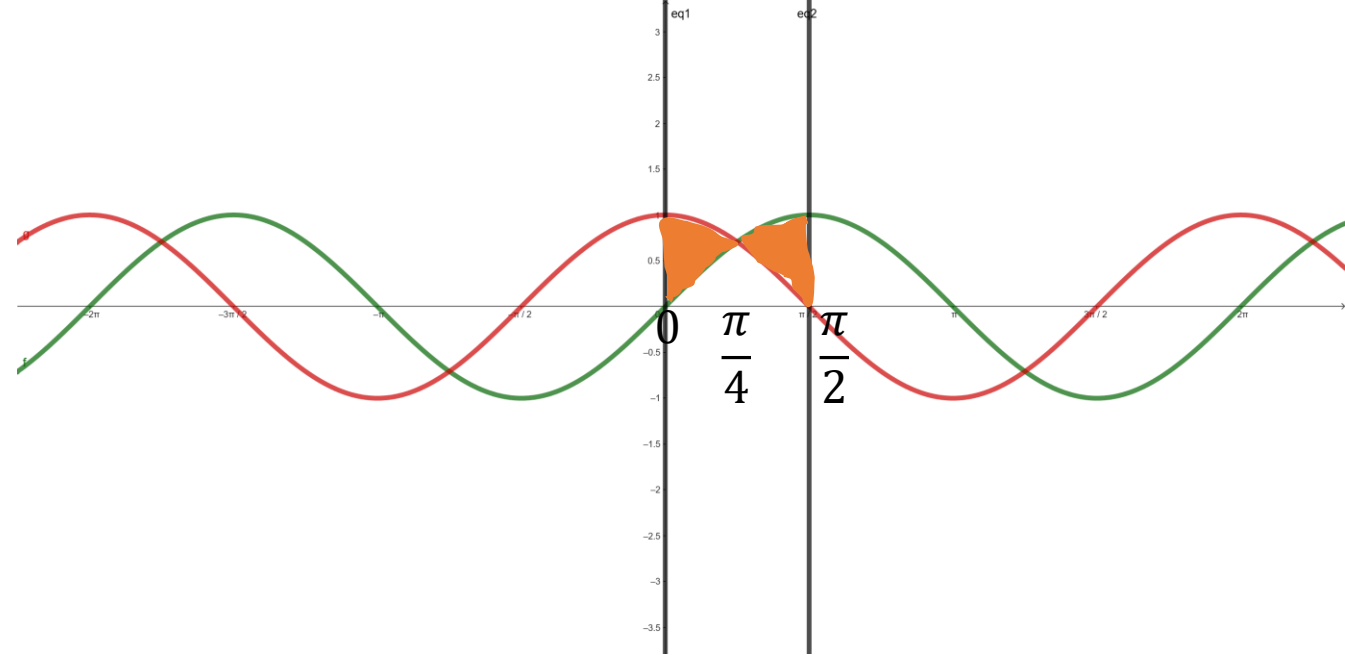
$$= x(\ln x)^2 - 2x \ln x + 2x$$

$$\begin{vmatrix} (\ln x)^2 & x \\ 2 \ln x \cdot \frac{1}{x} & 1 \end{vmatrix}$$

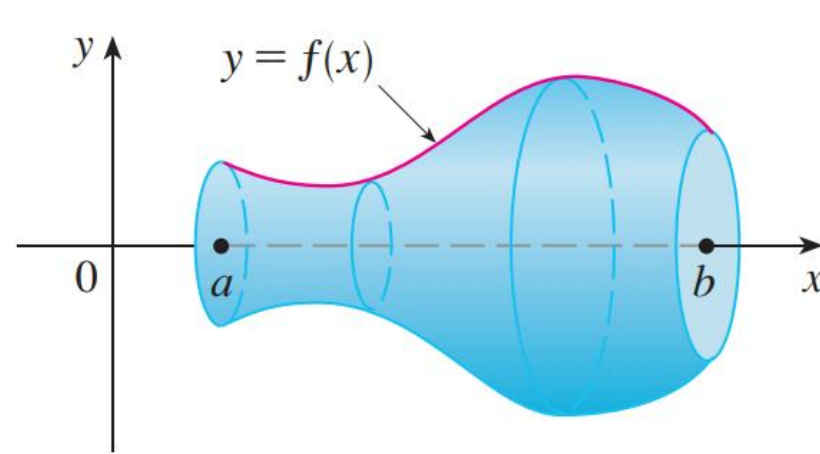
Spočtete objem tělesa, které vznikne rotací kolem osy x plochy ohraničené křivkami $y = \sin x$ a $y = \cos x$ a svislými přímkami $x = 0$ a $x = \frac{\pi}{2}$. Nakreslete si obrázek!

$$V = 2\left(\int_0^{\frac{\pi}{4}} \pi \cos^2 x \, dx - \int_0^{\frac{\pi}{4}} \pi \sin^2 x \, dx\right)$$

$$= 2\pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x) \, dx = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \, dx = 2\pi \left. \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{4}} = \pi \left(\sin \frac{\pi}{2} - \sin 0\right) = \pi$$



Povrch rotačního tělesa (Obsah pláště)



$$S = 2\pi \cdot \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2} dx$$

Vypočítejte obsah rotační plochy, která vznikne rotací dané křivky kolem osy x .

$$y = x^3, \quad 0 \leq x \leq 1 \quad S = 2\pi \cdot \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2} dx$$

$$= 2\pi \cdot \int_0^1 x^3 \sqrt{1 + [3x^2]^2} dx = 2\pi \cdot \int_0^1 x^3 \sqrt{1 + 9x^4} dx$$

$$1 + 9x^4 = u \quad 36x^3 dx = du$$

$$x^3 dx = \frac{1}{36} du$$

x	0	1
u	1	10

$$= 2\pi \cdot \frac{1}{36} \int_1^{10} \sqrt{u} du = \frac{\pi}{18} \int_1^{10} u^{\frac{1}{2}} du$$

$$= \frac{\pi}{18} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{10} = \frac{\pi}{27} (10\sqrt{10} - 1)$$

Goniometrické substituce

$$\int \sqrt{1-x^2} dx \quad x = \sin \theta \quad dx = \cos \theta d\theta$$

$$\int \frac{1}{\sqrt{-x^2-2x+1}} dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{1-\left(\frac{x+1}{\sqrt{2}}\right)^2}} dx$$
$$\frac{x+1}{\sqrt{2}} = \sin \theta \quad \frac{1}{\sqrt{2}} dx = \cos \theta d\theta$$

$$\int \sqrt{1+x^2} dx \quad x = \operatorname{tg} \theta \quad dx = \frac{1}{\cos^2 \theta} d\theta$$

$$\int \frac{1}{\sqrt{x^2-2x+4}} dx = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{1+\left(\frac{x-1}{\sqrt{3}}\right)^2}} dx$$
$$\frac{x-1}{\sqrt{3}} = \operatorname{tg} \theta \quad \frac{1}{\sqrt{3}} dx = \frac{1}{\cos^2 \theta} d\theta$$

Eulerova substituce

$$\sqrt{1-x^2} = xt + 1$$

$$\sqrt{-x^2-2x+1} = xt + 1$$

$$\sqrt{1+x^2} = x + t$$

$$\sqrt{x^2-2x+4} = x + t$$

Goniometrické substituce

$$\int \sqrt{x^2 - 1} dx \quad x = \frac{1}{\cos \theta} \quad dx = \frac{\sin \theta}{\cos^2 \theta} d\theta$$

$$\int \frac{x}{\sqrt{x^2 + 4x + 3}} dx = \int \frac{x}{\sqrt{(x + 2)^2 - 1}} dx$$

$$x + 2 = \frac{1}{\cos \theta} \quad dx = \frac{\sin \theta}{\cos^2 \theta} d\theta$$

Eulerovy substituce

$$\sqrt{x^2 - 1} = x + t$$

$$\sqrt{x^2 + 4x + 3} = x + t$$

$$\int_0^{0,6} \frac{x^2}{\sqrt{9-25x^2}} dx = \int_0^{0,6} \frac{x^2}{\sqrt{9(1-\frac{25}{9}x^2)}} dx = \frac{1}{3} \int_0^{0,6} \frac{x^2}{\sqrt{1-(\frac{5x}{3})^2}} dx$$

$$\frac{5x}{3} = \sin \theta \quad x = \frac{3}{5} \sin \theta$$

$$dx = \frac{3}{5} \cos \theta d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{(\frac{3}{5} \sin \theta)^2}{\sqrt{1-\sin^2 \theta}} \cdot \frac{3}{5} \cos \theta d\theta = \frac{9}{125} \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta \cos \theta}{|\cos \theta|} d\theta$$

x	0	0,6
θ	0	$\frac{\pi}{2}$

$$0 \leq x \leq \frac{\pi}{2} \Rightarrow |\cos \theta| = \cos \theta$$

$$= \frac{9}{125} \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta = \frac{9}{125} \int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2} d\theta$$

$$= \frac{9}{250} \int_0^{\frac{\pi}{2}} (1-\cos 2\theta) d\theta = \frac{9}{250} \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^{\frac{\pi}{2}} = \frac{9}{250} \left[\left(\frac{\pi}{2} - 0 \right) - 0 \right] = \frac{9\pi}{500}$$

$$\int_0^1 \sqrt{1+t^2} dt \quad t = \operatorname{sh}x \quad dt = \operatorname{ch}x dx \quad t = \frac{e^x - e^{-x}}{2} \quad 2t = e^x - e^{-x}$$

$$\ln(1+\sqrt{2}) \quad \int_0^{\ln(1+\sqrt{2})} \sqrt{1+sh^2x} \operatorname{ch}x dx \quad \operatorname{ch}^2x - \operatorname{sh}^2x = 1 \quad 2te^x = (e^x)^2 - 1$$

$$\Rightarrow 1 + \operatorname{sh}^2x = \operatorname{ch}^2x \quad (e^x)^2 - 2t(e^x) - 1 = 0$$

$$= \int_0^{\ln(1+\sqrt{2})} \operatorname{ch}^2x dx = \int_0^{\ln(1+\sqrt{2})} \left(\frac{e^x + e^{-x}}{2}\right)^2 dx$$

t	0	1
x	0	$\ln(1 + \sqrt{2})$

$$= \int_0^{\ln(1+\sqrt{2})} \frac{e^{2x} + 2 + e^{-2x}}{4} dx = \frac{1}{4} \left(\frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x} \right) \Big|_0^{\ln(1+\sqrt{2})}$$

$$= \frac{1}{4} \left(\frac{1}{2} (1 + \sqrt{2})^2 + 2 \ln(1 + \sqrt{2}) - \frac{1}{2} (1 + \sqrt{2})^{-2} \right) - \frac{1}{4} \left(\frac{1}{2} - \frac{1}{2} \right) \quad 0$$