Předmět: NMTM102 Matematická analýza II

Typ výuky: Cvičení

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Spočtěte určitý integrál $\int_1^4 \frac{\sqrt{x-1}}{x+2}dx = \int_0^{\sqrt{3}} \frac{y}{1+y^2+2} 2y dy = 2\int_0^{\sqrt{3}} \frac{y^2}{y^2+3} dy$

$y = \sqrt{x-1}$

$y^2 = x - 1 \Rightarrow 2y dy = dx$

$x = 1 + y^2$

x	1	4
y	0	$\sqrt{3}$

$$= 2\int_0^{\sqrt{3}} \frac{y^2+3-3}{y^2+3} dy = 2\int_0^{\sqrt{3}} \left(1 - \frac{3}{y^2+3}\right) dy = 2y\Big]_0^{\sqrt{3}} - 6\int_0^{\sqrt{3}} \frac{dy}{3\left(1+\frac{y^2}{3}\right)}$$

$$= 2\sqrt{3} - 2\int_0^{\sqrt{3}} \frac{dy}{1+\left(\frac{y}{\sqrt{3}}\right)^2} = 2\sqrt{3} - 2\, arctg\left(\frac{y}{\sqrt{3}}\right) . \sqrt{3}\,\Big]_0^{\sqrt{3}}$$

$$= 2\sqrt{3} - 2\sqrt{3}\left(\underbrace{arctg(1)}_{\frac{\pi}{4}} - \underbrace{arctg(0)}_{0} \right) = 2\sqrt{3}\left(1 - \frac{\pi}{4}\right)$$

Nějaké aplikace:

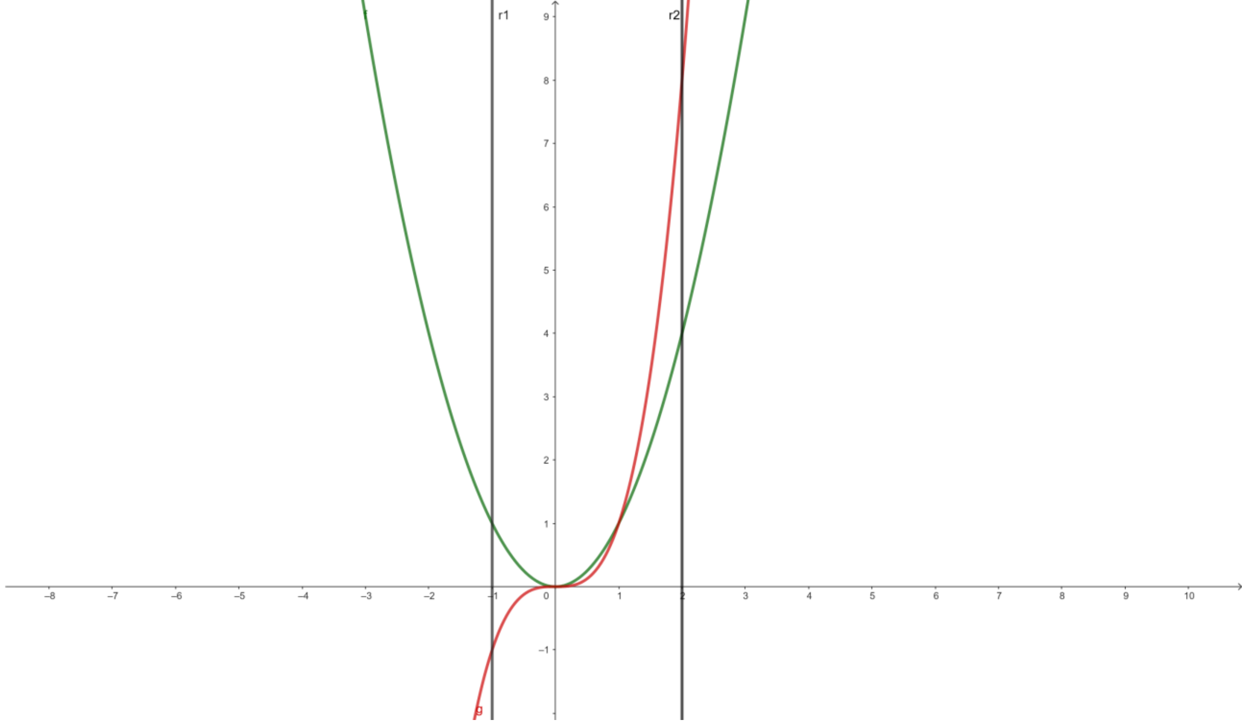
- Obsah
- Délka křivky $(l(f,[a,b]) = \int_a^b \sqrt{1+(f'(x))^2}\,dx,\ L = \int_\alpha^\beta \sqrt{[x'(t)]^2+[y'(t)]^2}\,dt)$
- Objem rotačního tělesa $(V = \int_a^b \pi(f(x))^2\,dx)$
- Povrch rotačního tělesa $(S = 2\pi.\int_a^b f(x).\sqrt{1+[f'(x)]^2}dx)$

Spočtěte velikost plochy ohraničené křivkami

$$y = x^2 \text{ a } y = x^3$$

a svislými přímkami $x = -1$ a $x = 2$.

Nakreslete si obrázek!



$$S = \int_{-1}^{1} (x^2 - x^3)dx + \int_{1}^{2} (x^3 - x^2)dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{-1}^{1} + \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_{1}^{2}$$

$$= \left(\frac{1}{3} - \frac{1}{4} - \left(-\frac{1}{3} - \frac{1}{4}\right)\right) + \left(\frac{16}{4} - \frac{8}{3} - \left(\frac{1}{4} - \frac{1}{3}\right)\right) = \frac{25}{12}$$

Spočtěte velikost plochy ohraničené křivkami

$$y = sin^2 x,\ \ \mathrm{y} = cos^2 x,\ \ x = -\frac{\pi}{4}, x = \frac{\pi}{4}$$

$$-\frac{\pi}{4} \leq x \leq \frac{\pi}{4} \Rightarrow cos^2 x \geq sin^2 x$$

$$S = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (cos^2 x - sin^2 x)\, dx\ = 2\int_0^{\frac{\pi}{4}} (cos^2 x - sin^2 x)\ dx$$

$$= 2\int_0^{\frac{\pi}{4}} \cos 2x\ dx = \sin 2x\ \Big]_0^{\frac{\pi}{4}} = \sin\frac{\pi}{2} - \sin 0 = 1$$

Spočtěte délku křivky

$$y = x^{\frac{3}{2}}, \qquad 0 \le x \le 1$$

Vzorec pro délku grafu funkce f na intervalu $[a,b]$: $l(f,[a,b]) = \int_a^b \sqrt{1+(f'(x))^2}\, dx$

Délka:

$$\int_0^1 \sqrt{1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\, dx = \int_0^1 \sqrt{1+\frac{9}{4}x}\, dx = \int_1^{\frac{13}{4}} \sqrt{y}\,\frac{4}{9}\, dy = \frac{4}{9}\int_1^{\frac{13}{4}} y^{\frac{1}{2}}\, dy$$

$$y = 1+\frac{9}{4}x \qquad dy = \frac{9}{4}dx$$

$$= \frac{4}{9}\left[\frac{2}{3}y^{\frac{3}{2}}\right]_1^{\frac{13}{4}}$$

$$dx = \frac{4}{9}dy$$

$$= \frac{8}{27}\left(\left(\frac{13}{4}\right)^{\frac{3}{2}} - 1\right)$$

x	0	1
y	1	$\frac{13}{4}$

Spočtěte délku křivky:

$$x = 1 + 3t^2, \quad y = 4 + 2t^3, 0 \leq t \leq 1.$$

$$L = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$x'(t) = 6t \qquad y'(t) = 6t^2$$

$$L = \int_0^1 \sqrt{(\ 6t\ )^2 + (\ 6t^2\ )^2}\, dt = \int_0^1 \sqrt{36t^2 + 36t^4}\ dt = \int_0^1 6t\sqrt{1 + t^2}\, dt$$

$$= 3\int_0^1 2t\sqrt{1 + t^2}\ dt = 3\int_1^2 \sqrt{u}\ du = 3\left[\ \frac{2}{3}u^{\frac{3}{2}}\ \right]_1^2 = \left[\ 2u\sqrt{u}\ \right]_1^2 = 4\sqrt{2} - 2$$

$$1 + t^2 = u$$

$$2t\,dt = du$$

t	0	1
u	1	2

Spočtěte délku křivky

$$y = 1 - \ln(\cos x)\,, \qquad 0 \le x \le \frac{\pi}{4}$$

Vzorec pro délku grafu funkce f na intervalu $[a,b]$:

$$l(f,[a,b]) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Délka:

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \left(-\frac{1}{\cos x}(-\sin x)\right)^2}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \frac{sin^2 x}{cos^2 x}}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{\frac{cos^2 x + sin^2 x}{cos^2 x}}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{|\cos x|}\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos x}\, dx = \int_0^{\frac{\pi}{4}} \frac{\cos x}{cos^2 x}\, dx = \int_0^{\frac{\pi}{4}} \frac{\cos x}{1 - sin^2 x}\, dx$$

$$\sin x = y$$
$$\cos x\, dx = dy$$

$$= \int_0^{\frac{\sqrt{2}}{2}} \frac{dy}{1 - y^2} = \int_0^{\frac{\sqrt{2}}{2}} \left(\frac{\cancel{A}\ \tfrac{1}{2}}{1+y} + \frac{\cancel{B}\ \tfrac{1}{2}}{1-y}\right) dy = \left[\frac{1}{2}\ln|1+y| - \frac{1}{2}\ln|1-y|\right]_0^{\frac{\sqrt{2}}{2}}$$

x	0	$\frac{\pi}{4}$
y	0	$\frac{\sqrt{2}}{2}$

$$= \left[\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right|\right]_0^{\frac{\sqrt{2}}{2}} = \frac{1}{2}\left(\ln\left|\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right| - \cancel{\ln 1}\ 0\right)$$

Spočtěte délku křivky: $x = t\sin t,\ y = t\cos t,\ 0 \le t \le 1.$

$$L = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

$x'(t) = \sin t + t\cos t \quad y'(t) = \cos t - t\sin t$

$$L = \int_0^1 \sqrt{[\sin t + t\cos t\ ]^2 + [\cos t - t\sin t\ ]^2}\,dt$$

$$L = \int_0^1 \sqrt{\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t}\ dt = \int_0^1 \sqrt{1+t^2}\,dt$$

$1 + t^2(\cos^2 t + \sin^2 t)$

$\frac{\pi}{4}$

$ch^2x - sh^2x = 1 \Rightarrow 1 + sh^2x = ch^2x$

$$\int_0^1 \sqrt{1+t^2}\,dt = \int_0^{\ln(1+\sqrt{2})} \sqrt{1+sh^2x}\,chx\,dx = \int_0^{\ln(1+\sqrt{2})} ch^2x\,dx = \int_0^{\ln(1+\sqrt{2})} \left(\frac{e^x+e^{-x}}{2}\right)^2 dx$$

$$= \int_0^{\ln(1+\sqrt{2})} \frac{e^{2x}+2+e^{-2x}}{4}\,dx = \frac{1}{4}\left(\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right)\Big]_0^{\ln(1+\sqrt{2})}$$

0

$$= \frac{1}{4}\left(\frac{1}{2}(1+\sqrt{2})^2 + 2\ln(1+\sqrt{2}) - \frac{1}{2}(1+\sqrt{2})^{-2}\right) - \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}\right)$$

$t = shx \quad dt = chxdx$

$t = \dfrac{e^x - e^{-x}}{2}$

$2t = e^x - e^{-x}$

$2te^x = (e^x)^2 - 1$

$(e^x)^2 - 2t(e^x) - 1 = 0$

t	0	1
x	0	$\ln(1+\sqrt{2})$