Předmět: NMTM102 Matematická analýza II

Typ výuky: Cvičení

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Cvičení 6 (25.3.2026)

$$\int \boldsymbol{R(\sin x\,, \cos x)dx}$$

6) $\displaystyle\int \frac{\sin x \cos x}{1 + sin^4x}\, dx$

1) $\displaystyle\int \frac{\cos x}{2 + \sin x} dx$

7) $\displaystyle\int \frac{dx}{(2 + \cos x) \sin x}$

2) $\displaystyle\int \frac{1}{1 + cos^2x} dx$

8) $\displaystyle\int \frac{1}{1 + \sin x + \cos x} dx$

3) $\displaystyle\int tgx\, dx$

9) $\displaystyle\int \frac{dx}{sin^2x + 2cos^2x}$

4) $\displaystyle\int tg^5x\, dx$

10) $\displaystyle\int \frac{tgx.(2 \cos x - \sin x)}{(cos^2x + 1).(\cos x + \sin x)} dx$

5) $\displaystyle\int \frac{sin^2x}{1 + sin^2x}\, dx$

$$\int R(\sin x, \cos x)dx = ?$$

I. $R(-\sin x, \cos x) = -R(sinx, cosx)$ $\boxed{\Rightarrow y = cos\,x}$

II. $R(\sin x, -\cos x) = -R(sinx, cosx)$ $\boxed{\Rightarrow y = sin\,x}$

III. $R(-\sin x, -\cos x) = R(sinx, cosx)$ $\boxed{\Rightarrow y = tg\,x}$

IV. $\boxed{y = tg(\frac{x}{2})}$

1) $\int \frac{\cos x}{2+\sin x} dx$

$$R(\sin x, \cos x) = \frac{\cos x}{2+\sin x}$$

$$R(\sin x, -\cos x) = \frac{-\cos x}{2+\sin x} = -R(\sin x, \cos x) \quad \boxed{\Rightarrow y = sin\, x}$$

$$dy = \cos x \, dx$$

$$\int \frac{\cos x}{2+\sin x} dx = \int \frac{dy}{2+y} \overset{c}{=} \ln|2+y| = \ln|2+\sin x|$$

2) $\int \frac{1}{1+cos^2 x} dx$

$R(\sin x, \cos x) = \frac{1}{1+cos^2 x}$ $\boxed{y = tg\, x}$ $\Rightarrow dy = \frac{1}{cos^2 x} dx$

$R(-\sin x, -\cos x) = R(sinx, cosx)$ **III.** $1 + tg^2 x = \frac{1}{cos^2 x} \rightarrow cos^2 x = \frac{1}{1+y^2}$

$$\int \frac{1}{1+cos^2 x} dx = \int \frac{cos^2 x}{1+cos^2 x} \frac{1}{cos^2 x} dx = \int \frac{cos^2 x}{1+cos^2 x} dy = \int \frac{\frac{1}{1+y^2}}{1+\frac{1}{1+y^2}} dy$$

$$\ldots = \int \frac{dy}{2+y^2} \quad \ldots \overset{c}{=} \frac{\sqrt{2}}{2} arctg(\frac{y}{\sqrt{2}}) = \frac{\sqrt{2}}{2} arctg(\frac{tgx}{\sqrt{2}})$$

$$3)\ \int tgx\, dx$$

I. $R(-\sin x\,,\cos x) = -R(sinx, cosx)$ $\boxed{\Rightarrow y = cos\, x}$

II. $R(\sin x\,,-\cos x) = -R(sinx, cosx)$ $\boxed{\Rightarrow y = sin\, x}$

III. $R(-\sin x\,,-\cos x) = R(sinx, cosx)$ $\boxed{\Rightarrow y = tg\, x}$

$$\int tgx\,dx = \int \frac{\sin x}{\cos x}dx$$

I. $R(-\sin x, \cos x) = -R(sinx, cosx)$ $\boxed{\Rightarrow y = cos\,x}$

$$dy = -\sin x\,dx$$

$$= -\int \frac{dy}{y} \overset{c}{=} -\ln|y| = -\ln|\cos x|$$

II. $R(\sin x, -\cos x) = -R(sinx, cosx)$ $\boxed{\Rightarrow y = sin\,x}$ $dy = cos\,x\,dx$

$$\int tgx\,dx = \int \frac{\sin x}{\cos x}dx = \int \frac{\sin x\,.\cos x}{\cos x\,.\cos x}dx = \int \frac{y}{1-y^2}dy \overset{c}{=} -\frac{1}{2}\ln|1-y^2|$$

$$= -\frac{1}{2}\ln|1-sin^2x|$$

III. $R(-\sin x, -\cos x) = R(sinx, cosx)$ $\boxed{\Rightarrow y = tg\,x}$ $dy = \frac{1}{cos^2x}dx$

$$\int tgx\,dx = \int y\,cos^2x\,dy$$

$$1 + tg^2x = \frac{1}{cos^2x} \rightarrow cos^2x = \frac{1}{1+y^2}$$

$$= \int \frac{y}{1+y^2}dy \overset{c}{=} \frac{1}{2}\ln|1+y^2| = \frac{1}{2}\ln|1+tg^2x|$$

4) $\int tg^5x\, dx$ **I, II, III** $\boxed{y = sinx, \quad y = cosx, \quad y = tg\, x}$

5) $\int \frac{sin^2x}{1 + sin^2x}\, dx$ **III.** $R(-\sin x, -\cos x) = R(sinx, cosx)$ $\boxed{y = tg\, x}$

6) $\int \frac{\sin x \cos x}{1 + sin^4x}\, dx$ **I, II, III** $\boxed{y = sinx, \quad y = cosx, \quad y = tg\, x}$

7) $\int \frac{dx}{(2 + \cos x)\sin x}$ **I.** $R(-\sin x, \cos x) = -R(sinx, cosx)$ $\boxed{y = \cos x}$

$$\int \frac{dx}{sin^2x + 2cos^2x} = \int \frac{dx}{cos^2x\ (\frac{sin^2x}{cos^2x} + 2)} = \int \frac{dx}{cos^2x\,(tg^2x \ \ +2)}$$

$$y = \operatorname{tg} x$$

$$\Rightarrow dy = \frac{1}{cos^2x} dx$$

$$= \int \frac{dy}{y^2 + 2} = \frac{1}{2} \int \frac{dy}{1 + (\frac{y}{\sqrt{2}})^2} \overset{c}{=} \frac{1}{2} arctg(\frac{y}{\sqrt{2}}) . \sqrt{2}$$

$$2(\frac{y^2}{2} + 1)$$

$$= \frac{\sqrt{2}}{2} arctg(\frac{\operatorname{tg} x}{\sqrt{2}})$$

$$\int \frac{tgx.(2\cos x - \sin x)}{(cos^2x+1).(\cos x+\sin x)}dx = \int \frac{tgx.(2\cos x-\sin x)\ cos^2x}{(cos^2x+1).(\cos x+\sin x)}\frac{1}{cos^2x}\ dx$$

$$= \int \frac{2\sin x\, cos^2x - sin^2x\cos x}{(cos^2x+1)\ (\cos x+\sin x)}\frac{1}{cos^2x}\ dx \quad = \int \frac{\frac{2\sin x\,cos^2x}{cos^3x} - \frac{sin^2x\cos x}{cos^3x}}{\frac{(cos^2x+1)}{cos^2x}\ \frac{(\cos x+\sin x)}{\cos x}}\frac{1}{cos^2x}\ dx$$

$$=\int\frac{2tgx\ - tg^2x}{\underset{(2+tg^2x)}{\cancel{(1+\frac{1}{cos^2x})}}(1+tgx)}\frac{1}{cos^2x}\ dx$$

$$y = tgx \quad dy = \frac{1}{cos^2x}dx$$

$$\frac{2y-y^2}{(2+y^2)(1+y)} = \frac{\overset{2}{\cancel{Ay+B}}}{2+y^2}+\frac{\overset{-1}{\cancel{C}}}{1+y}$$

$A = 0$

$B = 2$

$C = -1$

$$=\int\frac{2y\ -y^2}{(2+y^2)(1+y)}dy$$

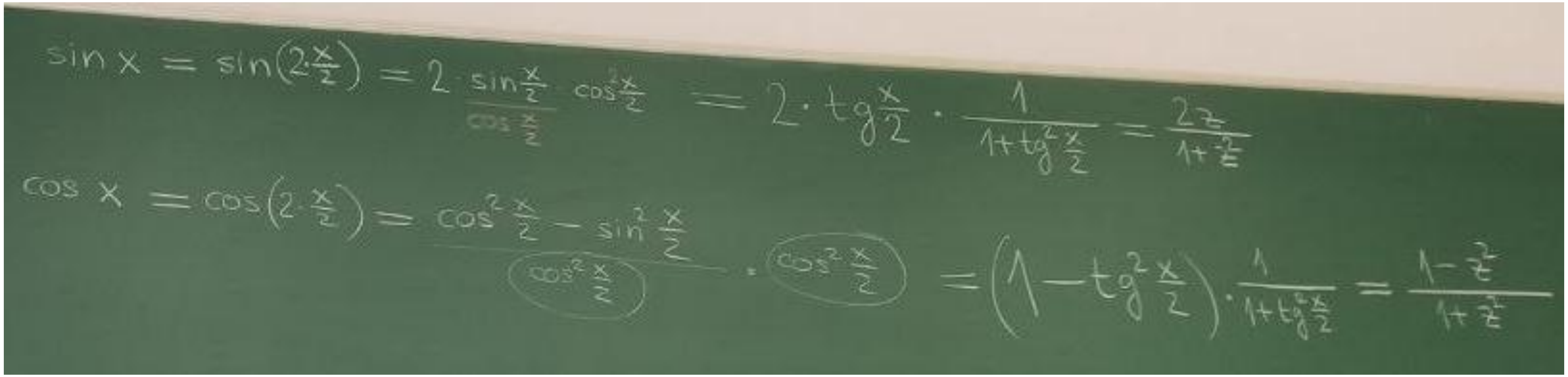
$$\sqrt{2}arctg(\frac{tgx}{\sqrt{2}}) - \ln|1+tgx|$$

$$=\int\frac{2}{\underset{2(1+\frac{y^2}{2})}{\cancel{2+y^2}}}dy - \int\frac{1}{1+y}dy = \int\frac{1}{1+(\frac{y}{\sqrt{2}})^2}dy - \int\frac{1}{1+y}dy \overset{c}{=} \cancel{\sqrt{2}arctg(\frac{y}{\sqrt{2}})} - \cancel{\ln|1+y|}$$

$$\int \frac{1}{1+\sin x+\cos x}dx \qquad t = tg(\frac{x}{2}) \qquad \sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

$$\frac{x}{2} = arctgt \qquad x = 2.arctgt \qquad dx = 2.\frac{1}{1+t^2}dt$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\, 2.\frac{1}{1+t^2}dt = \int \frac{1}{1+t}dt \stackrel{c}{=} \ln|1+t| = \ln|1+tg(\frac{x}{2})|$$



Určitý integrál

$$\int_a^b f(x)\ dx = F(x)\Big]_a^b = F(b) - F(a)$$

$$\int_0^1 x^2\ dx = \frac{x^3}{3}\Big]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

Spočtěte určitý integrál $\int_1^4 \frac{\sqrt{x-1}}{x+2}dx \;=\int_0^{\sqrt{3}} \frac{y}{1+y^2+2}2ydy \;= 2\int_0^{\sqrt{3}} \frac{y^2}{y^2+3}dy$

$y = \sqrt{x-1}$

$y^2 = x - 1 \Rightarrow 2ydy = dx$

$x = 1 + y^2$

x	1	4
y	0	$\sqrt{3}$

$$= 2\int_0^{\sqrt{3}} \frac{y^2+3-3}{y^2+3}dy \;= 2\int_0^{\sqrt{3}} (1 - \frac{3}{y^2+3})dy \;= 2y\Big]_0^{\sqrt{3}} - 6\int_0^{\sqrt{3}} \frac{dy}{3(1+\frac{y^2}{3})}$$

$$= 2\sqrt{3}\; - 2\int_0^{\sqrt{3}} \frac{dy}{1+(\frac{y}{\sqrt{3}})^2} = 2\sqrt{3}\; - 2\, arctg(\frac{y}{\sqrt{3}}) \cdot \sqrt{3}\;\Big]_0^{\sqrt{3}}$$

$$= 2\sqrt{3}\; - 2\sqrt{3}\;(\underbrace{\cancel{arctg}(1)}^{\frac{\pi}{4}} - \underbrace{\cancel{arctg}(0)}^{0}) = 2\sqrt{3}\,(1 - \frac{\pi}{4})$$