Předmět: NMTM102 Matematická analýza II

Typ výuky: Cvičení

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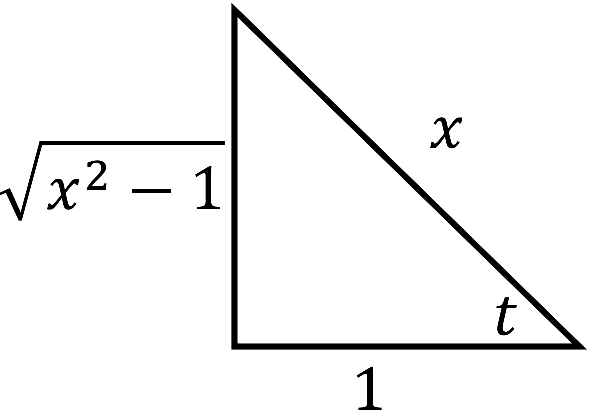
Cvičení 5 (18.3.2026)

$$\int \frac{dx}{(x^2-1)^{\frac{3}{2}}} \qquad x = \frac{1}{cost} \Rightarrow dx = \frac{-1}{cos^2t}(-sint)dt = \frac{sint}{cos^2t}dt$$

$$= \int \frac{1}{\left(\left(\frac{1}{cost}\right)^2 - 1\right)^{\frac{3}{2}}} \frac{sint}{cos^2t}dt = \int \left(\frac{cos^2t}{sin^2t}\right)^{\frac{3}{2}} \frac{sint}{cos^2t}dt = \int \frac{cos^3t}{sin^3t}\frac{sint}{cos^2t}dt$$

$$= \int \frac{cost}{sin^2t}\,dt = \int \frac{dy}{y^2} \overset{c}{=} -\frac{1}{y} = -\frac{1}{sint} = -\frac{x}{\sqrt{x^2-1}}$$

$$y = sint \Rightarrow dy = costdt$$



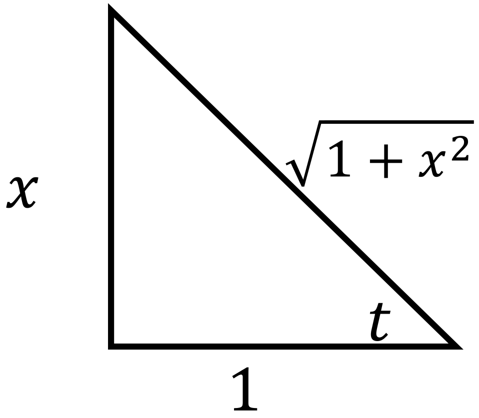
$$\cos t = \frac{1}{x} \qquad \sin t = \frac{\sqrt{x^2-1}}{x}$$

$$\int \frac{dx}{\sqrt{1+x^2}} \qquad x = tgt \;\Rightarrow dx = \frac{1}{cos^2 t} dt$$

$$= \int \frac{\frac{1}{cos^2 t} dt}{\sqrt{1+tg^2 t}} = \int \frac{\frac{1}{cos^2 t}}{\sqrt{\frac{1}{cos^2 t}}} dt = \int \frac{\frac{1}{cos^2 t}}{\frac{1}{cost}} dt = \int \frac{cost}{cos^2 t} dt = \int \frac{cost}{1 - sin^2 t} dt$$

$$y = sint \;\Rightarrow dy = costdt$$

$$= \int \frac{dy}{1-y^2} = \cdots \overset{c}{=} \frac{1}{2} \ln|1-y^2| = \frac{1}{2} \ln|1 - sin^2 t| = \frac{1}{2} \ln|cos^2 t| = \frac{1}{2} \ln\left|\frac{1}{1+x^2}\right|$$



$$tgt = x = \frac{x}{1} \qquad \cos t = \frac{1}{\sqrt{1+x^2}}$$

Eulerovy substituce: $\int R\left(x, \sqrt{ax^2+bx+c}\right) dx\,, \quad a, b, c \in \mathbb{R}, \quad a \neq 0, \quad b^2 - 4ac \neq 0$

1. Eulerova substituce: $a > 0 \qquad \sqrt{ax^2+bx+c} = \pm\sqrt{a}x + t \qquad x = \cdots \qquad dx = \cdots dt$

$\int \frac{dx}{x + \sqrt{x^2+x+1}} \qquad \sqrt{x^2+x+1} = -x + t \qquad \int \frac{dx}{\sqrt{x^2+x+1}} \qquad \sqrt{x^2+x+1} = x + t$

2. Eulerova substituce: $c > 0 \qquad \sqrt{ax^2+bx+c} = xt \pm \sqrt{c} \qquad x = \cdots \qquad dx = \cdots dt$

$\int \frac{dx}{1 + \sqrt{1-2x-x^2}} \qquad \sqrt{1-2x-x^2} = xt - 1 \qquad \int \frac{dx}{\sqrt{9-4x-x^2}} \qquad \sqrt{9-4x-x^2} = xt + 3$

3. Eulerova substituce: ax^2+bx+c roz.kv.trojčlen $\quad x_1, x_2 \qquad \sqrt{ax^2+bx+c} = t(x - x_1)$

nebo

$\sqrt{ax^2+bx+c} = t(x - x_2)$

$\int \frac{x - \sqrt{-x^2+3x-2}}{x + \sqrt{-x^2+3x-2}} dx \qquad x_1 = 1, \quad x_2 = 2 \qquad \sqrt{-x^2+3x-2} = t(x-1)$

nebo

$\sqrt{-x^2+3x-2} = t(x-2)$

Poznámka. Znaménko u $\sqrt{a}$ v 1. Eulerově substituci a u $\sqrt{c}$ ve 2. Eulerově substituci volíme většinou s přihlédnutím k tvaru funkce $R\left(x, \sqrt{ax^2 + bx + c}\right)$, ale v podstatě lze volit libovolně.

$$\int \frac{dx}{x+\sqrt{x^2+x+1}} \qquad a=1>0 \quad \sqrt{x^2+x+1}=-x+t \qquad x=\frac{t^2-1}{1+2t} \qquad dx=2\frac{t^2+t+1}{(1+2t)^2}dt$$

$$=\int \frac{2\frac{t^2+t+1}{(1+2t)^2}dt}{x-x+t} = 2\int \frac{t^2+t+1}{t\,(1+2t)^2}\,dt \qquad \frac{t^2+t+1}{t(1+2t)^2}=\frac{A}{t}+\frac{B}{1+2t}+\frac{C}{(1+2t)^2}$$

$$=2\int \frac{1}{t}dt-3\int \frac{1}{1+2t}dt-3\int \frac{1}{(1+2t)^2}dt \qquad t^2+t+1=A\,(1+2t)^2+B\,t\,(1+2t)+C\,t$$

$$\overset{c}{=} 2\ln|t|-\frac{3}{2}\ln|1+2t|+\frac{3}{2}\frac{1}{1+2t} \qquad t=x+\sqrt{x^2+x+1}$$

$t=0$ $\quad A=1$

$t=-\frac{1}{2}$ $\quad C=-\frac{3}{2}$

$t=1$ $\quad B=-\frac{3}{2}$

$$=2\ln\left|x+\sqrt{x^2+x+1}\right|-\frac{3}{2}\ln\left|1+2\left(x+\sqrt{x^2+x+1}\right)\right|+\frac{3}{2}\cdot\frac{1}{1+2\left(x+\sqrt{x^2+x+1}\right)}$$

$$\int \frac{dx}{\sqrt{1+x+x^2}} \qquad a = 1 > 0 \qquad \sqrt{1+x+x^2} = x + t \qquad x = \frac{t^2-1}{1-2t} \qquad dx = 2\frac{t-t^2-1}{(1-2t)^2}dt$$

$$= \int \frac{2\frac{t-t^2-1}{(1-2t)^2}dt}{\frac{t^2-1}{1-2t}+t} = \int \frac{2}{1-2t}dt \overset{c}{=} -\ln|1-2t| = -\ln|1-2(\sqrt{1+x+x^2}-x)|$$

$$t = \sqrt{1+x+x^2} - x$$

$$\int \frac{dx}{1+\sqrt{1-2x-x^2}} \qquad c = 1 > 0 \quad \sqrt{1-2x-x^2} = xt - 1$$

$$x = 2\frac{t-1}{t^2+1} \qquad dx = 2\frac{1+2t-t^2}{(t^2+1)^2}dt$$

$$= \int \frac{2\frac{1+2t-t^2}{(t^2+1)^2}dt}{\cancel{1} + xt - \cancel{1}} = \int \frac{2\frac{1+2t-t^2}{(t^2+1)^2}dt}{2\frac{t-1}{t^2+1}\,t} = -\int \frac{t^2-2t-1}{t(t-1)(t^2+1)}dt$$

$$= -\int \frac{1}{t}dt + \int \frac{1}{t-1}dt - \int \frac{2}{1+t^2}dt \qquad \frac{t^2-2t-1}{t(t-1)(t^2+1)} = \frac{A}{t} + \frac{B}{t-1} + \frac{Ct+D}{t^2+1}$$

$$\overset{c}{=} -\ln|t| + \ln|t-1| - 2arctgt \qquad t^2-2t-1 = A(t-1)(t^2+1) + Bt(t^2+1) + (Ct+D)t(t-1)$$

$$t = \frac{\sqrt{1-2x-x^2}+1}{x}$$

$$t=0 \quad A=1$$

$$t=1 \quad B=-1$$

$$t=-1 \quad C=0$$

$$t=2 \quad D=2$$

$$= -\ln\left|\frac{\sqrt{1-2x-x^2}+1}{x}\right| + \ln\left|\frac{\sqrt{1-2x-x^2}+1}{x} - 1\right| - 2arctg\left(\frac{\sqrt{1-2x-x^2}+1}{x}\right)$$

$$\int \frac{dx}{\sqrt{1-4x-x^2}} \qquad c = 1 > 0 \quad \sqrt{1-4x-x^2} = xt + 1 \quad x = \frac{-4-2t}{1+t^2} \quad dx = 2\frac{t^2+4t-1}{(t^2+1)^2}dt$$

$$= \int \frac{2\frac{t^2+4t-1}{(t^2+1)^2}dt}{\frac{t^2-1}{1-2t}t + 1} = \int \frac{-2}{t^2+1}dt \overset{c}{=} -2arctgt \; = -2arctg\left(\frac{\sqrt{1-4x-x^2}-1}{x}\right)$$

$$t = \frac{\sqrt{1-4x-x^2}-1}{x}$$

$$\int \frac{x-\sqrt{-x^2+3x-2}}{x+\sqrt{-x^2+3x-2}}dx$$

$$-x^2+3x-2=0$$

$$x_1=1 \quad x_2=2$$

$$-x^2+3x-2=-(x-1)(x-2)$$

$$\sqrt{-x^2+3x-2}=(x-1)t$$

$$x=\frac{t^2+2}{1+t^2}$$

$$dx=-\frac{2t}{(1+t^2)^2}dt$$

$$=\int \frac{\frac{t^2+2}{1+t^2}-(x-1)t}{\frac{t^2+2}{1+t^2}+(x-1)t}\left(-\frac{2t}{(1+t^2)^2}\right)dt$$

$$=\int \frac{\frac{t^2+2}{1+t^2}-\left(\frac{t^2+2}{1+t^2}-1\right)t}{\frac{t^2+2}{1+t^2}+\left(\frac{t^2+2}{1+t^2}-1\right)t}\left(-\frac{2t}{(1+t^2)^2}\right)dt=-2\int \frac{t^3-t^2+2t}{(t^2+t+2)(1+t^2)^2}dt=\cdots$$