Předmět: NMTM102 Matematická analýza II

Typ výuky: Cvičení

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Cvičení 4 (11.3.2026)

$$\int \frac{x^3}{x^3-2x^2+x-2}\,dx \qquad x^3 : x^3-2x^2+x-2$$

$$= \int \frac{x^3-2x^2+x-2 \;\; -(-2x^2+x-2)}{x^3-2x^2+x-2}\,dx \;=\; \int \left(1-\frac{-2x^2+x-2}{x^3-2x^2+x-2}\right)dx$$

$$\frac{-2x^2+x-2}{x^3-2x^2+x-2} = \frac{-2x^2+x-2}{(x-2)(x^2+1)} = \frac{\overset{-\frac{8}{5}}{\cancel{A}}}{(x-2)} + \frac{\overset{-\frac{2}{5}}{\cancel{B}}x+\overset{\frac{1}{5}}{\cancel{C}}}{(x^2+1)}$$

$$-2x^2+x-2 = A(x^2+1)+(Bx+C)(x-2) \;=\; Ax^2+A+Bx^2-2Bx+Cx-2C$$

$$= (A+B)x^2+x(-2B+C)+A-2C$$

$$A+B=-2 \qquad -2B+C=1 \qquad A-2C=-2 \qquad A=-\frac{8}{5} \quad B=-\frac{2}{5} \quad C=\frac{1}{5}$$

$$= x-\int\left(\frac{-\frac{8}{5}}{x-2}+\frac{-\frac{2}{5}x+\frac{1}{5}}{x^2+1}\right)dx = x+\frac{8}{5}\ln|x-2|+\frac{1}{5}\int\frac{2x}{x^2+1}\,dx-\frac{1}{5}\int\frac{dx}{x^2+1}$$

$$\overset{c}{=} x+\frac{8}{5}\ln|x-2|+\frac{1}{5}\ln(x^2+1)-\frac{1}{5}\,arctg x$$

$$\int \frac{x+7}{(x+7)^2+1}dx = \frac{1}{2}\int \frac{2(x+7)}{(x+7)^2+1}dx \overset{c}{=} \frac{1}{2}\ln|(x+7)^2+1| = \frac{1}{2}\ln((x+7)^2+1)$$

$$\int \frac{x+7}{(x+6)^2}dx = \int \frac{(x+6)+1}{(x+6)^2}dx = \int \frac{x+6}{(x+6)^2}dx + \int \frac{1}{(x+6)^2}dx$$

$$= \int \frac{1}{x+6}dx + \int (x+6)^{-2}\,dx$$

$$\overset{c}{=} \ln|x+6| + \frac{(x+6)^{-1}}{-1}$$

$$\int \frac{x+2}{(x-1)^2(x+1)}\,dx$$

$$\frac{x+2}{(x-1)^2(x+1)} = \frac{\overset{-\frac{1}{4}}{\cancel{A}}}{x-1} + \frac{\overset{\frac{3}{2}}{\cancel{B}}}{(x-1)^2} + \frac{\overset{\frac{1}{4}}{\cancel{C}}}{x+1}$$

$$x+2 = A(x-1)(x+1) + B(x+1) + C(x-1)^2 = Ax^2 - A + Bx + B + Cx^2 - 2Cx + C$$

x^2: $0 = A + C$ $\qquad C = \frac{1}{4}$

x: $1 = B - 2C$ $\qquad A = -\frac{1}{4}$

x^0: $2 = -A + B + C$ $\qquad B = \frac{3}{2}$

$$= \int \left(\frac{-\frac{1}{4}}{x-1} + \frac{\frac{3}{2}}{(x-1)^2} + \frac{\frac{1}{4}}{x+1} \right) dx$$

$$\overset{c}{=} -\frac{1}{4}\ln|x-1| + \frac{3}{2} \cdot \frac{(x-1)^{-1}}{-1} + \frac{1}{4}\ln|x+1|$$

$$\int \frac{x^5}{x^2+x-2}\,dx$$

$$x^5 : x^2+x-2 \quad = x^3 - x^2 + 3x - 5 + \frac{11x-10}{x^2+x-2}$$

$$\vdots$$

$$\overline{11x-10}$$

$$\frac{32}{3}\ln|x+2| + \frac{1}{3}\ln|x-1|$$

$$= \int \left(x^3 - x^2 + 3x - 5 + \frac{11x-10}{x^2+x-2}\right) dx \overset{c}{=} \frac{x^4}{4} - \frac{x^3}{3} + 3\frac{x^2}{2} - 5x + \int \frac{11x-10}{x^2+x-2}\,dx$$

$$\frac{11x-10}{(x+2)(x-1)} = \frac{\overset{\frac{32}{3}}{A}}{x+2} + \frac{\overset{\frac{1}{3}}{B}}{x-1}$$

$$11x - 10 = A(x-1) + B(x+2) \qquad A = \frac{32}{3} \quad B = \frac{1}{3}$$

$$\int \frac{dx}{x^6+1}$$

$$x^6+1 = (x^2)^3+1 = (x^2+1)(x^4-x^2+1)$$

$$A^3+B^3 = (A+B)(A^2-AB+B^2)$$

$$x^4-x^2+1 = x^4+1-x^2 = (x^2+1)^2-3x^2 = (x^2+1)^2-(\sqrt{3}x)^2$$

$$= (x^2+1-\sqrt{3}x)(x^2+1+\sqrt{3}x)$$

$$\frac{1}{x^6+1} = \frac{1}{(x^2+1)(x^2-\sqrt{3}x+1)(x^2+\sqrt{3}x+1)}$$

$$= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2-\sqrt{3}x+1} + \frac{Ex+F}{x^2+\sqrt{3}x+1}$$

$$\int \frac{dx}{(x^2+1)^2}$$

Per partes

$$I_n = \int \frac{dx}{(x^2+a^2)^n}$$

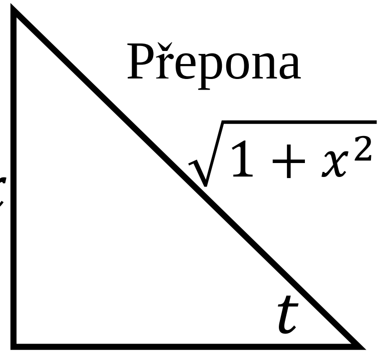
$$I_n = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}$$

$$\int \frac{dx}{(x^2+1)^2} \qquad x = tgt \;\Rightarrow dx = \frac{1}{cos^2 t}dt$$

$$= \int \frac{\frac{1}{cos^2 t}dt}{\cancel{(tg^2 t + 1)^2}} = \int cos^2 t\, dt = \int \frac{1 + cos2t}{2}dt \overset{c}{=} \frac{1}{2}\left(t + \frac{1}{2}sin2t\right)$$

$$\frac{1}{cos^4 t}$$

$$= \frac{1}{2}\left(arctgx + \frac{x}{1+x^2}\right)$$



$$sin2t = 2\sin t \cos t = \frac{2x}{1+x^2}$$

$$tgt = x = \frac{x}{1} \qquad \sin t = \frac{x}{\sqrt{1+x^2}} \qquad \cos t = \frac{1}{\sqrt{1+x^2}}$$

$$\int \frac{dx}{(x^2-1)^{\frac{3}{2}}} \qquad x = \frac{1}{cost} \Rightarrow dx = \frac{-1}{cos^2 t}(-sint)dt = \frac{sint}{cos^2 t}dt$$

$$\int \frac{dx}{(x^2-1)^{\frac{3}{2}}} = \dots = -\frac{x}{\sqrt{x^2-1}}$$

$$\int \frac{dx}{\sqrt{1+x^2}} \qquad x = tgt \;\Rightarrow dx = \frac{1}{cos^2t}dt$$

$$\int \frac{dx}{\sqrt{1+x^2}} \quad \ldots = \frac{1}{2}\ \ln\left|\frac{1}{1+x^2}\right|$$