Předmět: NMTM102 Matematická analýza II

Typ výuky: Cvičení

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Cvičení 3 (4.3.2026)

$$\int \frac{x^2}{cos^2(x^3)}dx = \int \frac{\frac{1}{3}dt}{cos^2(t)} \overset{c}{=} \frac{1}{3}\, tagt = \frac{1}{3}\, tag(x^3)$$

$$t = x^3 \Rightarrow dt = 3x^2dx$$

$$x^2dx = \frac{1}{3}dt$$

$$(tagx)' = \frac{1}{cos^2x}$$

$$\int \frac{dx}{x\, lnx} = \int \frac{dt}{t} \overset{c}{=} ln\,|t| = ln\,|\, lnx\,|$$

$$t = lnx \Rightarrow dt = \frac{1}{x}dx$$

$$\int \frac{x^3}{x^8+2}dx = \int \frac{x^3}{(x^4)^2+2}dx = \int \frac{\frac{1}{4}dt}{t^2+2} = \frac{1}{4}\int \frac{dt}{t^2+2} = \frac{1}{4}\int \frac{dt}{2(\frac{t^2}{2}+1)}$$

$$t = x^4 \Rightarrow dt = 4x^3dx$$

$$x^3dx = \frac{1}{4}dt$$

$$= \frac{1}{8}\int \frac{dt}{(\frac{t}{\sqrt{2}})^2+1} \overset{c}{=} \frac{1}{8}\arctan(\frac{t}{\sqrt{2}}) \cdot \sqrt{2}$$

$$= \frac{\sqrt{2}}{8}\arctan(\frac{x^4}{\sqrt{2}})$$

$$\int x^5 e^{x^3}\,dx = \int x^2 x^3 e^{x^3} dx = \frac{1}{3}\int t\ e^t\ dt \overset{c}{=} \frac{1}{3}(\ t\ e^t - e^t) = \frac{1}{3}(x^3 e^{x^3} - e^{x^3})$$

$$x^3 = t \quad 3x^2 dx = dt$$

$$x^2 dx = \frac{1}{3} dt$$

$$\begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix}$$

$$\int \frac{dx}{x \ln x \ln(\ln x)} = \int \frac{dt}{t \ \ln t} = \int \frac{du}{u} \overset{c}{=} \ln|u| = \ln|\ln t| = \ln|\ln(\ln x)|$$

$$\ln x = t \qquad\qquad \ln t = u$$

$$\frac{1}{x} dx = dt \qquad\qquad \frac{1}{t} dt = du$$

$$\int \frac{1}{3+4x^2}\,dx$$

$$(arctgx)' = \frac{1}{1+x^2}$$

$$\int \frac{dx}{1+x^2} \overset{c}{=} arctgx$$

$$= \int \frac{1}{3(1+\frac{4x^2}{3})}\,dx$$

$$= \frac{1}{3}\int \frac{1}{1+(\frac{2x}{\sqrt{3}})^2}\,dx \overset{c}{=} \frac{1}{3}\ arctg(\frac{2x}{\sqrt{3}}) \,.\, \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{6}\, arctg(\frac{2x}{\sqrt{3}})$$

$$\int \frac{1}{x^2+6x+10}dx = \int \frac{1}{(x+3)^2+1}dx \overset{c}{=} arctg(x+3)$$

$$\int \frac{1}{x^2+6x+11}dx = \int \frac{1}{(x+3)^2+2}dx = \int \frac{1}{2(1+\frac{(x+3)^2}{2})}dx$$

$$= \frac{1}{2}\int \frac{1}{1+(\frac{x+3}{\sqrt{2}})^2}dx \overset{c}{=} \frac{1}{2}\, arctg(\frac{x+3}{\sqrt{2}}).\sqrt{2} = \frac{\sqrt{2}}{2}\, arctg(\frac{x+3}{\sqrt{2}})$$

$$\int \frac{f'(x)}{f(x)}dx \quad = \int \frac{dy}{y} \overset{c}{=} \ln|y| = \ln|f(x)|$$

$$y = f(x) \Rightarrow dy = f'(x)dx$$

$$\int \frac{6x^5 + 24x^2}{x^6 + 8x^3}dx \overset{c}{=} \ln|\,x^6 + 8x^3\,|$$

$$\int \frac{3x+9}{x^2+6x+11}dx \quad = \int \frac{3(x+3)}{x^2+6x+11}dx$$

$$= \frac{3}{2}\int \frac{2(x+3)}{x^2+6x+11}dx = \frac{3}{2}\int \frac{2x+6}{x^2+6x+11}dx = \frac{3}{2}\int \frac{dy}{y} \overset{c}{=} \frac{3}{2}\ln|y| = \frac{3}{2}\ln|x^2+6x+11|$$

$$= \frac{3}{2}\ln(x^2+6x+11)$$

$$y = x^2+6x+11 \Rightarrow dy = (2x+6)\,dx$$

$$\int \frac{3x+6}{x^2+6x+11}dx = \frac{3}{2}\int \frac{2(x+2)}{x^2+6x+11}dx = \frac{3}{2}\int \frac{2x+4+2-2}{x^2+6x+11}dx$$

$$= \frac{3}{2}\int \frac{2x+6-2}{x^2+6x+11}dx = \frac{3}{2}\int \left(\frac{2x+6}{x^2+6x+11} - \frac{2}{x^2+6x+11}\right)dx$$

$$\overset{c}{=} \frac{3}{2}\left(\sout{\ln|x^2+6x+11|} - \sqrt{2}.arctg\left(\frac{x+3}{\sqrt{2}}\right)\right)$$

$$\ln(x^2+6x+11)$$

$$\int \frac{1}{\sqrt{2-3x^2}}dx = \int \frac{1}{\sqrt{2(1-\frac{3x^2}{2})}}dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{1-(\frac{\sqrt{3}x}{\sqrt{2}})^2}}dx \overset{c}{=} \frac{1}{\sqrt{2}} arcsin(\frac{\sqrt{3}x}{\sqrt{2}}) \cdot \frac{\sqrt{2}}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} arcsin(\sqrt{\frac{3}{2}}x)$$

$$(arcsinx)' = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{dx}{\sqrt{1-x^2}} \overset{c}{=} arcsinx$$

$$\int \frac{x^2}{x+1}\,dx$$

$$x^2 : x + 1 = x - 1$$

$$-(x^2 + x)$$

$$-x$$

$$-(-x - 1)$$

$$1$$

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

$$\int \frac{x^2}{x+1}\,dx = \int \left(x - 1 + \frac{1}{x+1}\right) dx \overset{c}{=} \frac{x^2}{2} - x + \ln|x+1|$$

$$\int \frac{dx}{3x^2-2x-1}$$

$3x^2-2x-1=0 \quad x=1 \quad x=\frac{-1}{3}$

$$3x^2-2x-1=3(x-1)(x+\frac{1}{3})=(x-1)(3x+1)$$

$$=\int \frac{dx}{(x-1)(3x+1)}$$

$$\frac{1}{(x-1)(3x+1)}=\frac{A}{x-1}+\frac{B}{3x+1} \qquad A=\frac{1}{4},\; B=-\frac{3}{4}$$

$$1=A(3x+1)+B(x-1)$$

$$x=1 \Rightarrow 1=4A \quad A=\frac{1}{4}$$

$$x=-\frac{1}{3} \Rightarrow 1=\frac{-4}{3}B \quad B=\frac{-3}{4}$$

$$=\frac{1}{4}\int \frac{dx}{x-1}-\frac{3}{4}\int \frac{dx}{3x+1}$$

$$\overset{c}{=}\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|3x+1|$$

$$=\frac{1}{4}(\ln|x-1|-\ln|3x+1|)$$

$$=\frac{1}{4}\ln\left|\frac{x-1}{3x+1}\right|$$

$$\int \frac{1}{x^2 + 4x + 5} dx = \int \frac{1}{(x+2)^2 + 1} dx \overset{c}{=} arctg(x+2)$$

$$\int \frac{x+1}{x^2+2x+9}dx = \frac{1}{2}\int \frac{2(x+1)}{x^2+2x+9}dx \overset{c}{=} \frac{1}{2}ln\,|x^2+2x+9|$$

$$\int \frac{lnx}{x\sqrt{1+lnx}}dx = \int \frac{y-1}{\sqrt{y}}\,dy = \int \left(\frac{y}{\sqrt{y}} - \frac{1}{\sqrt{y}} \right)dy = \int \left(\sqrt{y}\ [y^{\frac{1}{2}}] - y^{-\frac{1}{2}} \right)dy$$

$$y = 1 + lnx \Rightarrow dy = \frac{1}{x}dx$$

$$lnx = y - 1$$

$$\overset{c}{=} \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$= \frac{2}{3}y^{\frac{3}{2}} - 2y^{\frac{1}{2}}$$

$$= \frac{2}{3}(1+lnx)^{\frac{3}{2}} - 2(1+lnx)^{\frac{1}{2}}$$

$$\int \frac{x+1}{x^2+4}dx = \int \left(\frac{x}{x^2+4} + \frac{1}{x^2+4} \right) dx = \frac{1}{2}\int \frac{2x}{x^2+4}dx + \int \frac{1}{x^2+4}dx$$

$$4\left(1+\frac{x^2}{4}\right)$$

$$= \frac{1}{2} ln \mid x^2+4 \mid + \frac{1}{4}\int \frac{1}{1+(\frac{x}{2})^2} \, dx$$

$$\overset{c}{=} \frac{1}{2} ln(x^2+4) + \frac{1}{4} arctg(\frac{x}{2}) \; .2$$

$$= \frac{1}{2} ln(x^2+4) + \frac{1}{2} arctg(\frac{x}{2})$$

$$\int \frac{x^2}{(1-x)^{100}}\,dx = \int \frac{(1-y)^2}{y^{100}}(-dy) = \int \frac{1-2y+y^2}{y^{100}}(-dy)$$

$$y = 1 - x \Rightarrow dy = -dx$$

$$x = 1 - y$$

$$= -\int \left(\frac{1}{y^{100}} - \frac{2}{y^{99}} + \frac{1}{y^{98}}\right) dy$$

$$= -\int \left(y^{-100} - 2y^{-99} + y^{-98}\right) dy$$

$$\overset{c}{=} -\left(\frac{y^{-99}}{-99} - 2\frac{y^{-98}}{-98} + \frac{y^{-97}}{-97}\right)$$

$$= -\left(\frac{(1-x)^{-99}}{-99} - 2\frac{(1-x)^{-98}}{-98} + \frac{(1-x)^{-97}}{-97}\right)$$

$$\int \frac{x^{17}-5}{x^2-1}dx$$

$$x^{17}-5 : x^2-1 = x^{15}+x^{13}+x^{11}+x^9+\dots+x = \sum_{k=1}^{8} x^{2k-1}$$

$$\begin{array}{l} -\;\; x^{17}-x^{15} \\ \hline x^{15}-5 \\ -\;\; x^{15}-x^{13} \\ \hline x^{13}-5 \\ \vdots \\ \hline x-5 \end{array}$$

$$\frac{x^{17}-5}{x^2-1} = \sum_{k=1}^{8} x^{2k-1} + \frac{x-5}{x^2-1}$$

$$\frac{x-5}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$x-5 = A(x+1) + B(x-1)$$

$$x=-1 \Rightarrow -6=-2B \qquad B=3$$

$$x=1 \Rightarrow -4=2A \qquad A=-2$$

$$= \int \left(\sum_{k=1}^{8} x^{2k-1} + \frac{x-5}{x^2-1}\right)dx$$

$$= \sum_{k=1}^{8} \int x^{2k-1}dx + \int \frac{x-5}{x^2-1}dx = \sum_{k=1}^{8} \frac{x^{2k}}{2k} + \int \frac{A}{x-1}dx + \int \frac{B}{x+1}dx + c$$

$$-2\ln|x-1| \qquad 3\ln|x+1|$$

$$\int \frac{1}{\sqrt{2-x^2}}dx \quad = \int \frac{1}{\sqrt{2(1-\frac{x^2}{2})}}dx$$

$$(arcsinx)' = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{dx}{\sqrt{1-x^2}} \overset{c}{=} arcsinx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{1-(\frac{x}{\sqrt{2}})^2}}dx \overset{c}{=} \frac{1}{\sqrt{2}}\, arcsin(\frac{x}{\sqrt{2}}) \,.\sqrt{2}$$

$$= arcsin(\frac{x}{\sqrt{2}})$$

$$\int \frac{1}{x^2 - 4x + 6} dx \quad = \int \frac{1}{(x-2)^2 + 2} dx \quad = \int \frac{1}{2(\frac{(x-2)^2}{2} + 1)} dx$$

$$= \frac{1}{2} \int \frac{1}{1 + (\frac{x-2}{\sqrt{2}})^2} dx \overset{c}{=} \frac{1}{2} arctg(\frac{x-2}{\sqrt{2}}) . \sqrt{2}$$

$$\int \frac{x}{x^2-4x+6}dx = \frac{1}{2}\int \frac{2\,x}{x^2-4x+6}dx = \frac{1}{2}\int \frac{2x -4+4}{x^2-4x+6}dx$$

$$= \frac{1}{2}\int (\frac{2x-4}{x^2-4x+6} + \frac{4}{x^2-4x+6})\,dx \stackrel{c}{=} \frac{1}{2}(\ln|x^2-4x+6|+2\sqrt{2}.arctg(\frac{x-2}{\sqrt{2}}))$$

$$\int \frac{3x+7}{x^2-2x-3}\,dx = \int \frac{3x+7}{(x-3)(x+1)}\,dx = \int \left(\frac{4}{x-3} + \frac{-1}{x+1}\right) dx \overset{c}{=} 4\ln|x-3| - \ln|x+1|$$

$$= \ln(x-3)^4 - \ln|x+1|$$

$$= \ln\left|\frac{(x-3)^4}{x+1}\right|$$

$$\frac{3x+7}{(x-3)(x+1)} = \frac{\overset{4}{\cancel{A}}}{x-3} + \frac{\overset{-1}{\cancel{B}}}{x+1} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

$$3x+7 = A(x+1) + B(x-3)$$

$$3x+7 = Ax + A + Bx - 3B$$

$$3x+7 = x(A+B) + A - 3B$$

$$A + B = 3$$

$$A - 3B = 7$$

$$B = -1$$

$$A = 4$$

$$\int \frac{dx}{x^2-1} = \int \left(\frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}\right) dx \overset{c}{=} \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| = \frac{1}{2}(\ln|x-1| - \ln|x+1|)$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|$$

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{\overset{\frac{1}{2}}{\cancel{A}}}{x-1} + \frac{\overset{-\frac{1}{2}}{\cancel{B}}}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

$$1 = A(x+1) + B(x-1)$$

$$1 = Ax + A + Bx - B$$

$$1 = x(A+B) + A - B$$

$$0x + 1 = x(A+B) + A - B$$

$$A + B = 0$$

$$A - B = 1$$

$$A = \frac{1}{2}$$

$$B = -\frac{1}{2}$$

$$\int \frac{e^x}{e^x+1} dx = \int \frac{dy}{y} \stackrel{c}{=} \ln|y| = \ln|\, e^x + 1\,| = \ln(e^x + 1)$$

$$y = e^x + 1 \Rightarrow dy = e^x\, dx$$

$$\int \sin(\ln x)\,dx = \int \sin(t)\,e^t dt \overset{c}{=} \frac{e^t \sin t - e^t \cos t}{2} = \frac{x \sin(\ln x) - x\cos(\ln x)}{2}$$

$$\ln x = t$$

$$x = e^t \Rightarrow dx = e^t dt$$

$$\int \frac{cosx}{e^x} dx = \int cosx\, e^{-x}\, dx = e^{-x} \sin x - \int sinx\,(-e^{-x}) dx$$

$$\begin{vmatrix} e^{-x} & \sin x \\ -e^{-x} & cosx \end{vmatrix} \qquad = e^{-x} \sin x + \int e^{-x} \sin x\, dx \qquad \begin{vmatrix} e^{-x} & -cosx \\ -e^{-x} & sinx \end{vmatrix}$$

$$= e^{-x} \sin x + (-e^{-x} \cos x - \int e^{-x} \cos x\, dx)$$

$$\int \frac{cosx}{e^x} dx = e^{-x} \sin x - e^{-x} \cos x - \int \frac{cosx}{e^x} dx$$

$$2 \int \frac{cosx}{e^x} dx \overset{c}{=} e^{-x} \sin x - e^{-x} \cos x \Rightarrow \int \frac{cosx}{e^x} dx \overset{c}{=} \frac{e^{-x} \sin x - e^{-x} \cos x}{2}$$

$$\int \frac{\ln x}{x\sqrt{1+\ln x}}\,dx \qquad 1+\ln x = t \qquad \frac{1}{x}dx = dt$$

$$\ln x = t-1$$

$$= \int \frac{t-1}{\sqrt{t}}dt = \int \frac{t}{\sqrt{t}} - \frac{1}{\sqrt{t}}\,dt = \int t^{\frac{1}{2}} - t^{-\frac{1}{2}}\,dt \overset{c}{=} \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^{\frac{1}{2}}}{\frac{1}{2}} = \frac{2}{3}(1+\ln x)^{\frac{3}{2}} - 2(1+\ln x)^{\frac{1}{2}}$$

$$\int arctgx\ dx = \int 1.arctgx\ dx = x\ arctgx - \frac{1}{2}\int \frac{2x}{1+x^2}\ dx \overset{c}{=} xarctgx - \frac{1}{2}\ln|1+x^2|$$

$$\left| \begin{matrix} arctgx & x \\ \frac{1}{1+x^2} & 1 \end{matrix} \right|$$

$$= xarctgx - \frac{1}{2}\ln(1+x^2)$$

$$\int \sqrt{x}\, arctg\sqrt{x}\, dx = \frac{2}{3}x^{\frac{3}{2}}\, arctg\sqrt{x} - \frac{2}{3}\int x\sqrt{x}\,\frac{1}{1+x}\,\frac{1}{2\sqrt{x}}\, dx$$

$$\left| \begin{array}{ll} arctg\sqrt{x} & \frac{2}{3}x^{\frac{3}{2}} \\ \frac{1}{1+x}\,\frac{1}{2\sqrt{x}} & \sqrt{x} \end{array} \right|$$

$$1 - \frac{1}{x+1}$$

$$= \frac{2}{3}x^{\frac{3}{2}} arctg\sqrt{x} - \frac{1}{3}\int \frac{x+1-1}{x+1}\, dx$$

$$\overset{c}{=} \frac{2}{3}x^{\frac{3}{2}} arctg\sqrt{x} - \frac{1}{3}(x - \ln|x+1|)$$

$$\int \frac{1}{2x^2+3}dx$$

$$(arctgx)' = \frac{1}{1+x^2}$$

$$= \int \frac{1}{3(\frac{2x^2}{3}+1)}dx$$

$$\int \frac{dx}{1+x^2} \stackrel{c}{=} arctgx$$

$$= \frac{1}{3}\int \frac{1}{1+(\sqrt{\frac{2}{3}}x)^2}dx \stackrel{c}{=} \frac{1}{3}arctg(\sqrt{\frac{2}{3}}x).\sqrt{\frac{3}{2}}$$

$$\int cot\, x\, dx = \int \frac{cosx}{sinx} dx = \int \frac{dy}{y} \overset{c}{=} \ln|y| = \ln|\,sinx\,|$$

$$y = sinx \Rightarrow dy = cosx\, dx$$

$$\int x^2 arctg(x+1)dx = \frac{x^3}{3} arctg(x+1) - \int \frac{x^3}{3} \frac{1}{1+(x+1)^2} dx$$

$$\left| \begin{matrix} arctg(x+1) & \frac{x^3}{3} \\ \frac{1}{1+(x+1)^2} & x^2 \end{matrix} \right|$$

$$= \frac{x^3}{3} arctg(x+1) - \frac{1}{3}\int \frac{x^3}{x^2+2x+2} dx$$

$$x^3 : x^2+2x+2 = x-2+\frac{2x+4}{x^2+2x+2}$$

$$= \frac{x^3}{3} arctg(x+1) - \frac{1}{3}\left(\int x-2\,dx + \int \frac{2x+4}{x^2+2x+2} dx\right)$$

$$= \frac{x^3}{3} arctg(x+1) - \frac{1}{3}\left(\frac{x^2}{2} - 2x + \int \frac{2x+2}{x^2+2x+2} dx + \int \frac{2}{x^2+2x+2} dx\right)$$

$$\ln(x^2+2x+2)$$

$$2\int \frac{1}{(x+1)^2+1} dx \quad 2arctg(x+1)$$

$$\overset{c}{=} \frac{x^3}{3} arctg(x+1) - \frac{1}{3}\left(\frac{x^2}{2} - 2x + \ln(x^2+2x+2) + 2arctg(x+1)\right)$$