Charles University

Faculty of Mathematics and Physics;

Departemnet of Mathematics Education

Developing conceptual knowledge in school mathematics

Lesson #4

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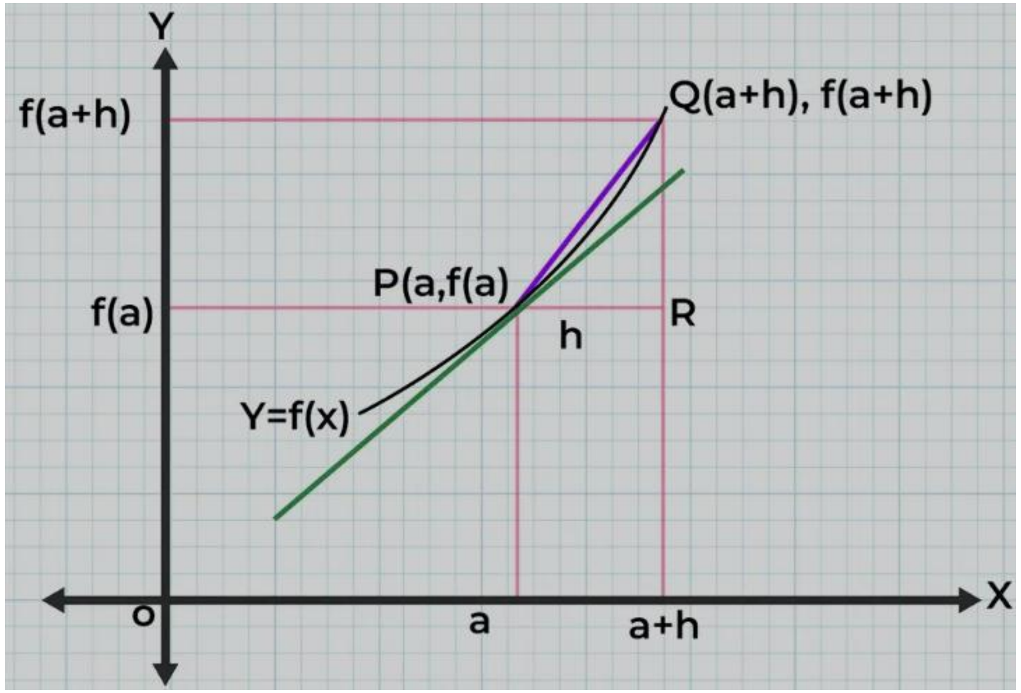
The concept of the derivative in different representations

❑ Students’ difficulties in learning the concept of the derivative

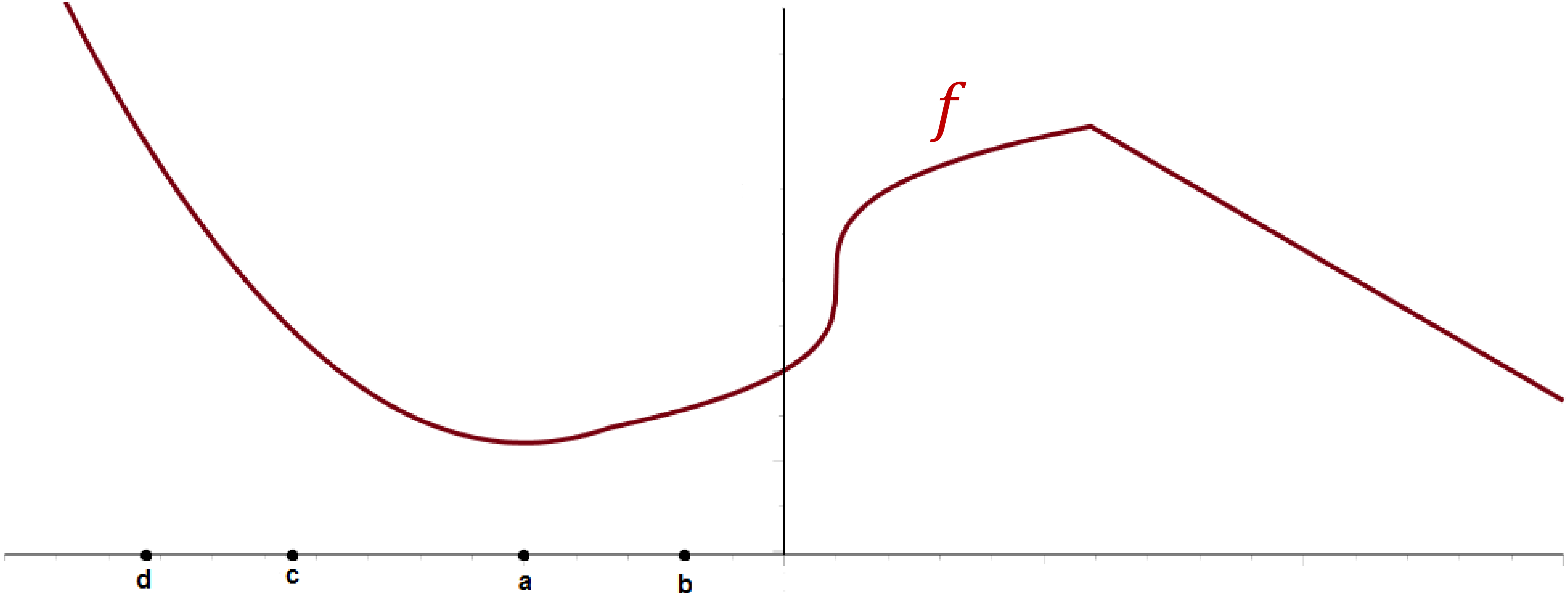
- Previous studies report that many students, when learning about derivatives, focus more on symbolic representations than on graphical representations (Biza, 2021; Ryberg, 2018), and they have difficulties establishing logical connections between these representations (Chang et al., 2016; Haghjoo & Reyhani, 2021; Zandieh, 2000).

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x = -\operatorname{cosec}^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$
$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$
$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$



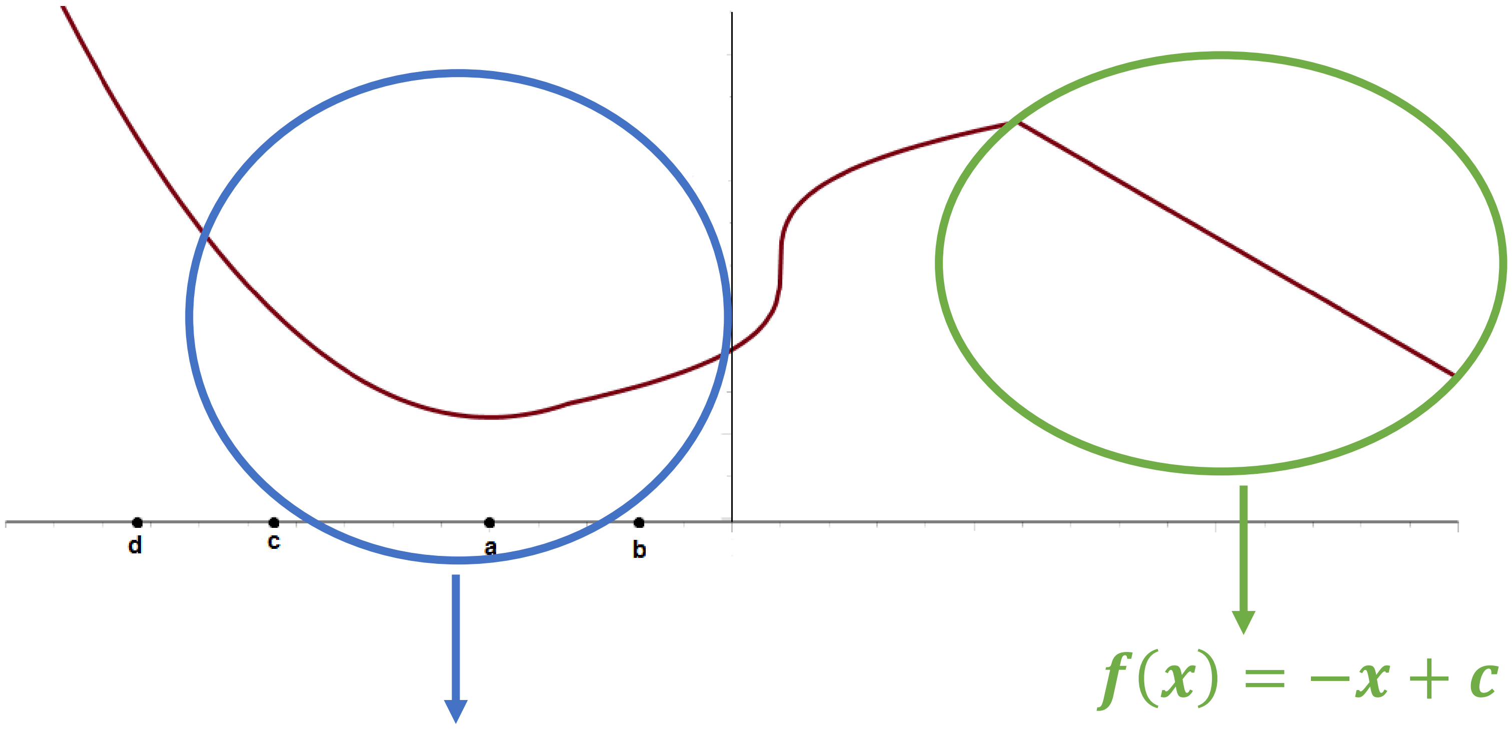
❑ Students' difficulties in learning the concept of the derivative

- Students often find it difficult to draw the graph of the derivative when they are given <u>only the graph of *f*</u>.



❑ Students' difficulties in learning the concept of the derivative

- To draw the graph of the derivative (f'), students often feel that they necessarily need the algebraic representation of the function (f) (Garcia-Garcia & Dolores-Flores, 2021).



$$f(x) = -x + c$$

$$f'(x) = -1$$

$$f(x) = x^2$$

$$f'(x) = 2x$$

❑ The concept of the derivative in different representations:

We can consider several representations of the concept of the derivative:

- ➢ Algebraic representation [Symbolic representation]
- ➢ Graphical representation
- ➢ Numerical representation

- **Algebraic representation [Symbolic representation]**

There are three different algebraic representations for the formal definition of the derivative of a function f at a point, which are used to compute the derivative at x_0:

$$\boldsymbol{f'(x_0) =}$$

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \qquad \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

We distinguish among these three definitions of the derivative at a particular point and consider each of them as a separate realization, because many students encounter difficulties in understanding how these three representations are related to one another.

➢ Graphical representation [Geometrical representation]

❖ Checking the existence of the derivative at a point by zooming in on the graph of the function near that point.

https://www.geogebra.org/graphing?lang=en

❖ $f(x) = x^2 \quad A[0,0]$ ✔

❖ $g(x) = \sin(x) \quad B[1, \sin(1)]$ ✔

❖ $h(x) = |x| \quad C[0,0]$ ✘

- **Graphical representation**

- Approximating the derivative at a point using the slopes of secant lines.

https://www.geogebra.org/m/ykzk8YgF

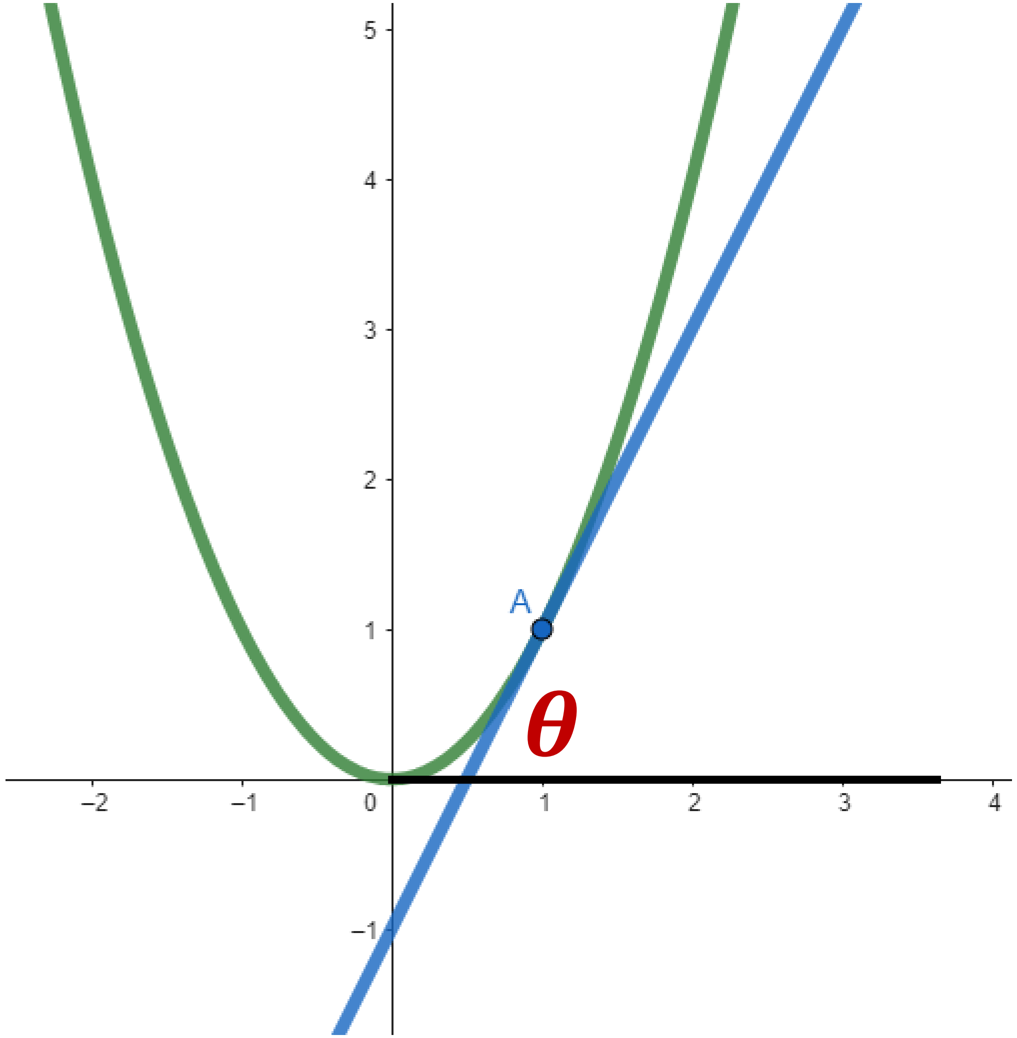
➢ Graphical representation

❖ Computing the derivative at a point as the limit of the slopes of secant lines.

https://www.geogebra.org/m/ykzk8YgF

➢ Graphical representation

❖ Computing the derivative at a point using $\tan\theta$ (where θ is the angle between the tangent line and the positive direction of the x -axis).



➢ Numerical representation

❖ Approximating the derivative at the point x_0 using a difference quotient,

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h} \qquad f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h} \qquad f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

but without applying the formal process $h \to 0$

- **Numerical representation**

Approximating the derivative at the point x_0 using a difference quotient.

x	3	3,2
$f(x)$	28	33,768

$$f'(3) \approx ?$$

$$f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h}$$

$$f'(3) \approx \frac{f(3,2)-f(3)}{3,2\ -3} = \frac{33,768-28}{0,2} = 28,84$$

$$\boxed{f'(3) \approx 28,84}$$

➢ Numerical representation

Approximating the derivative at a point using a table of values of the difference quotient as h approaches zero.

x	1,7	1,8	1,9	2	2,1	2,2	2,3
$f(x)$	4,59	5,04	5,51	6	6,51	7,04	7,59
$x-2$	$-0{,}3$	$-0{,}2$	$-0{,}1$		0,1	0,2	0,3
$f(x)-f(2)$	$-1{,}41$	$-0{,}96$	$-0{,}49$		0,51	1,04	1,59
$\frac{f(x)-f(2)}{x-2}$	4,7	4,8	4,9		5,1	5,2	5,3

$\boldsymbol{f'(2)} \approx ?$

→ **5** ←

Using the values of the continuous function $f(x)$ given in the table, approximate the value of its derivative at the point $x = 2$.

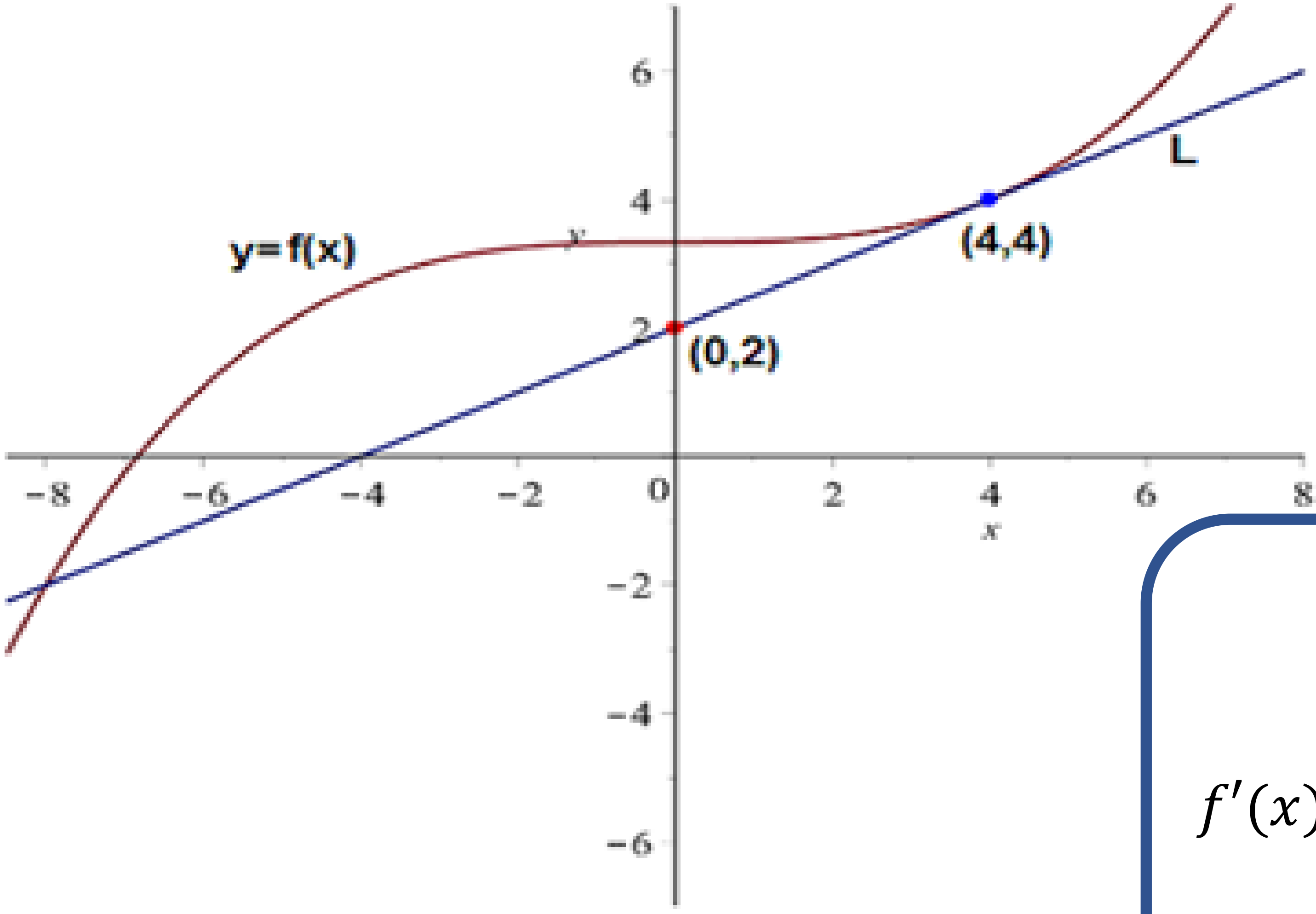
x	1,9	1,99	1,999	2	2,0001	2,001	2,1
$f(x)$	3,61	3,9601	3,996001	4	4,00040001	4,004001	4,41

$$f'(2) = ?$$

$\frac{f(x) - f(2)}{x - 2}$	3,9	3,99	3,999		4,0001	4,001	4,1

$$f'(2) = 4$$

Assume that the line L is tangent to the graph of the function f at the point (4,4), as shown in the figure. Find $f'(4)$.



$[4;\ 4]\quad [0;\ 2]$

$$m_L = \frac{4-2}{4-0} = \frac{1}{2}$$

$$y = \frac{1}{2}x + 2$$

$$f'(x) = (\frac{1}{2}x + 2)' = \frac{1}{2}$$

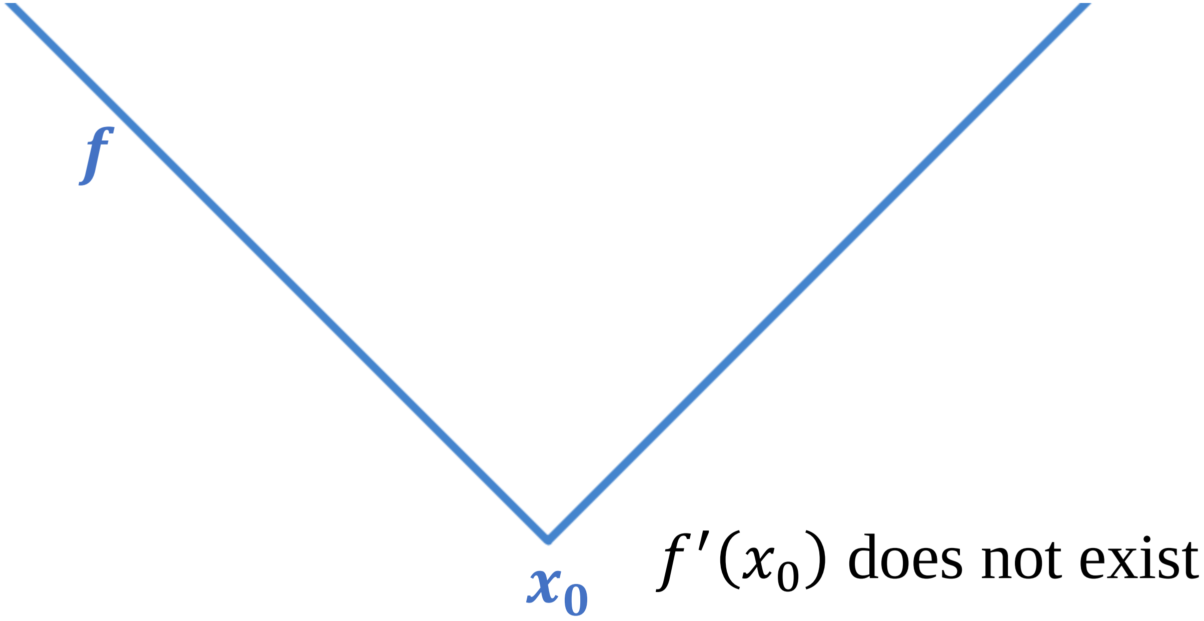
$$f'(4) = \frac{1}{2}$$

True or False?

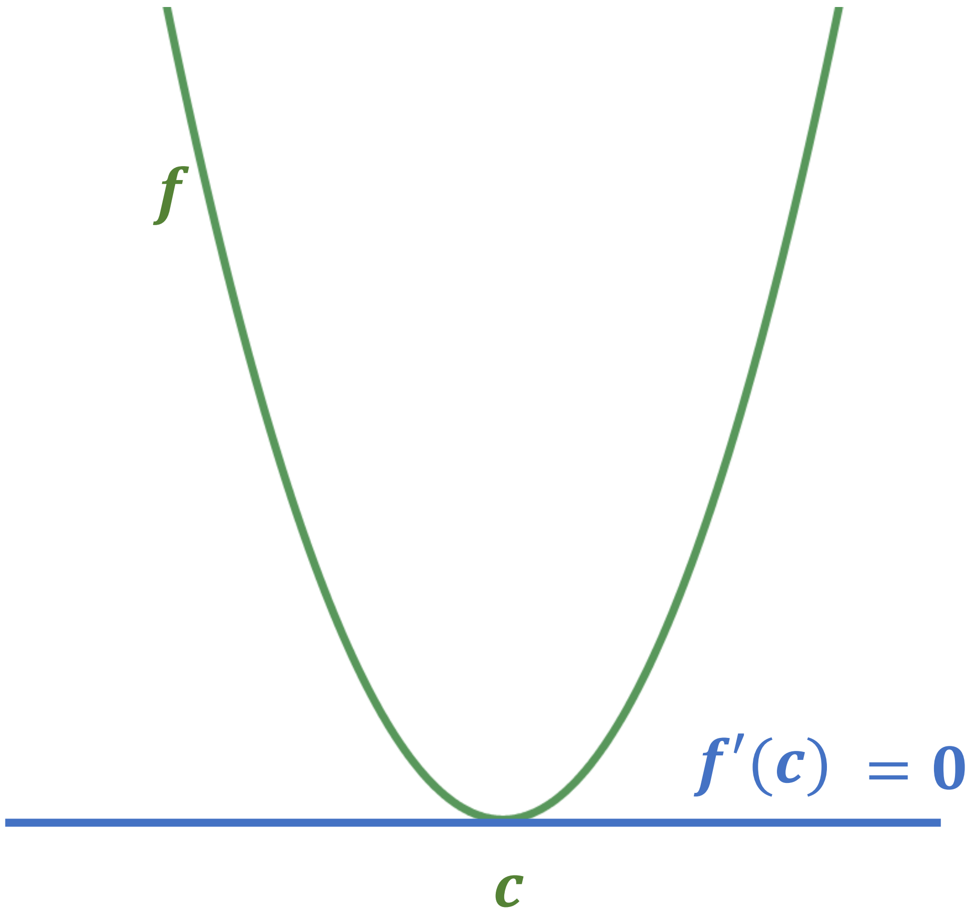
Statement 1:

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. If $(x_0, f(x_0))$ is a minimum point, then $f'(x_0) = 0$.

True / False

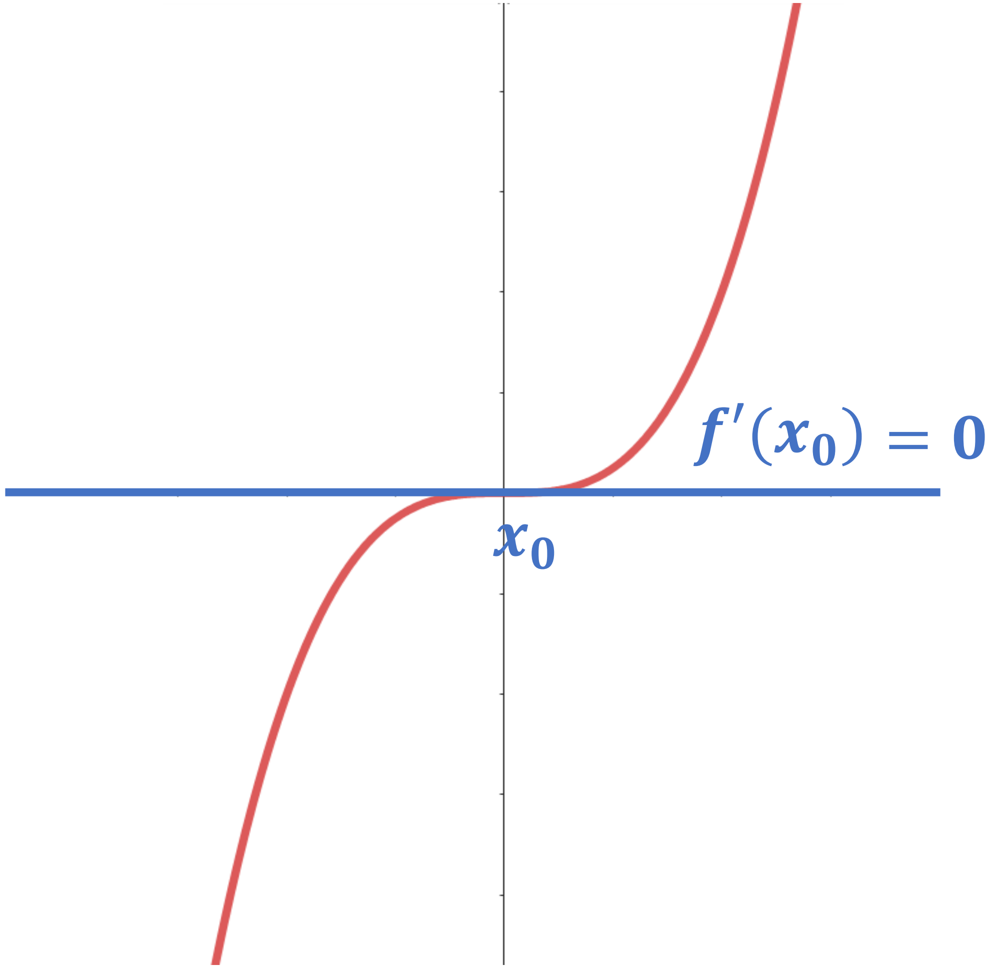


Let the function f have a local extremum at the point c. If $f'(c)$ exists, then $f'(c) = 0$.



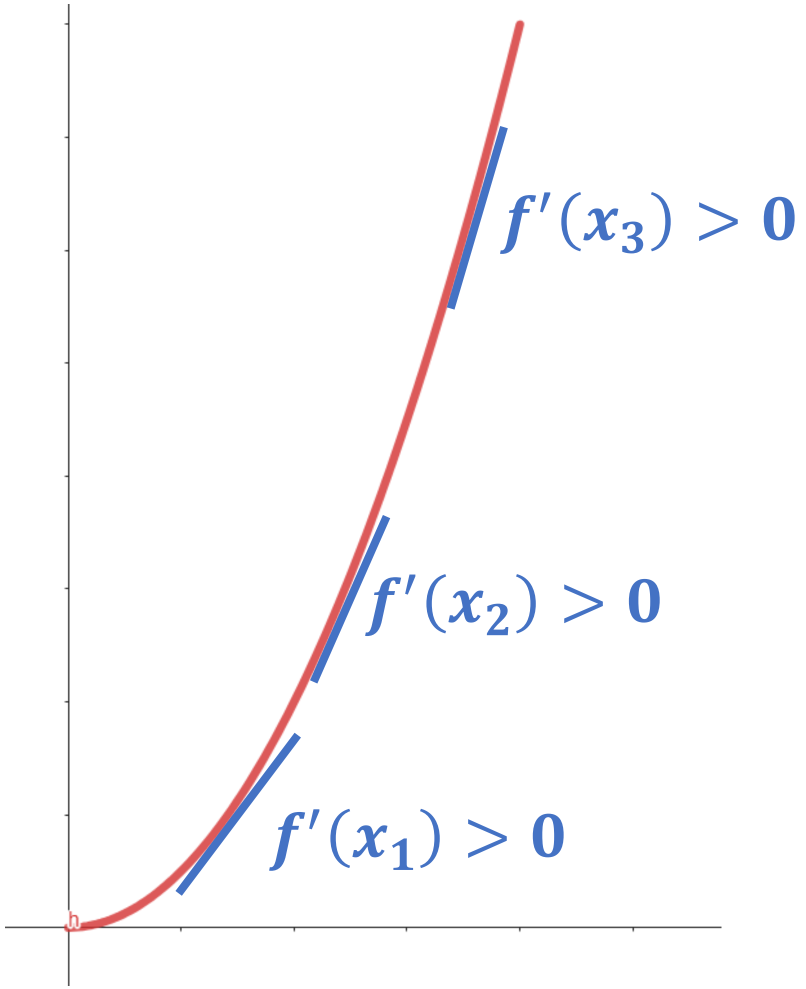
Statement 2: Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(x_0) = 0$, then $(x_0, f(x_0))$ is a minimum or maximum point.

True / False



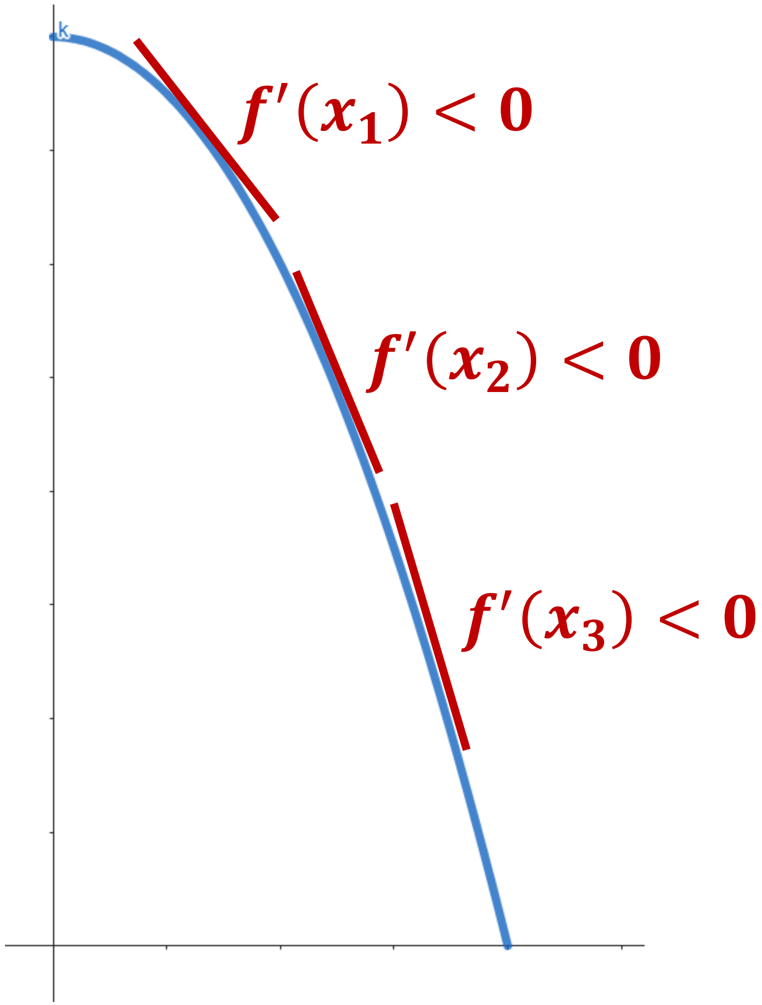
Let the function f be continuous on the interval $[a, b]$ and let $f'(x)$ exist for all $x \in (a, b)$. Then:

$$\forall x \in (a, b): f'(x) > 0 \Rightarrow f \text{ is increasing on } (a, b)$$

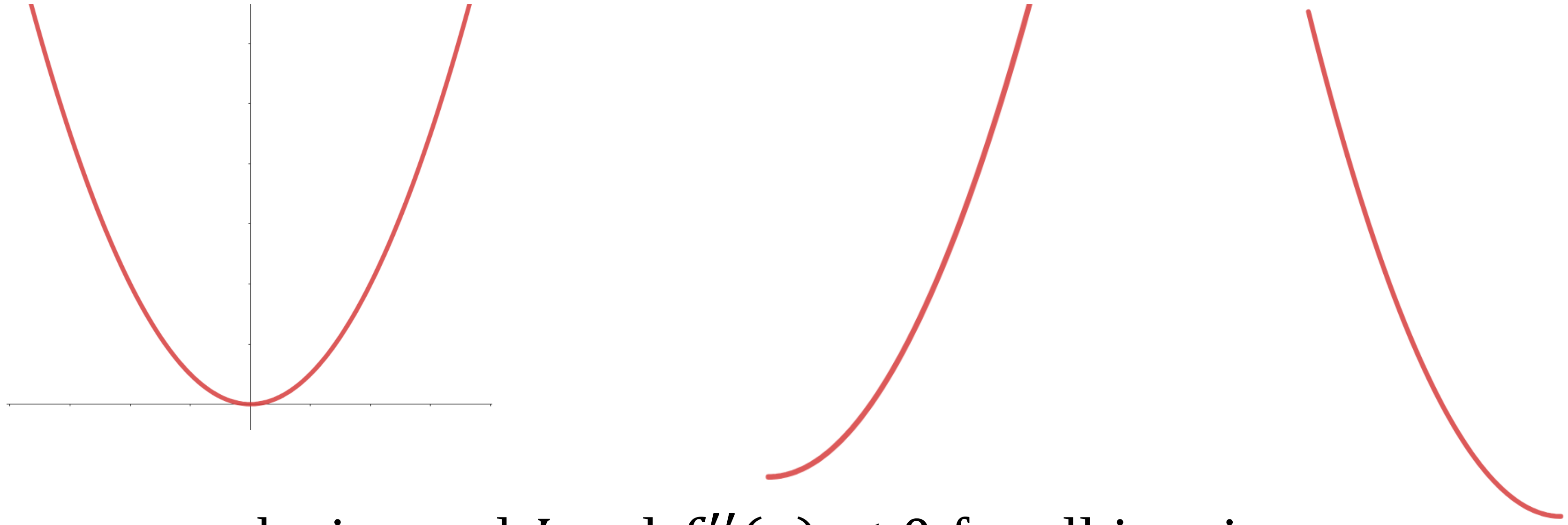


Let the function f be continuous on the interval $[a, b]$ and let $f'(x)$ exist for all $x \in (a, b)$. Then:

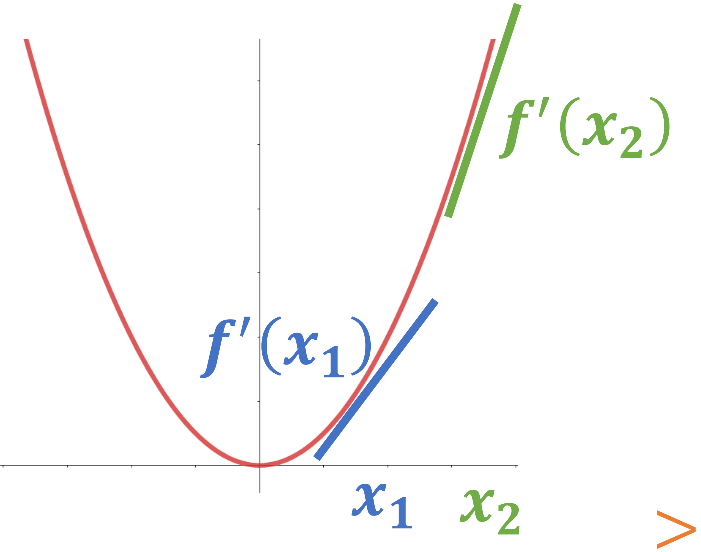
$$\forall x \in (a, b): f'(x) < 0 \Rightarrow f \text{ is decreasing on } (a, b)$$

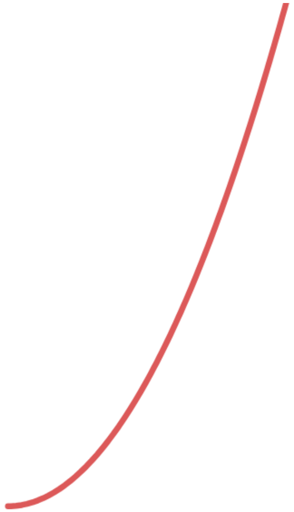


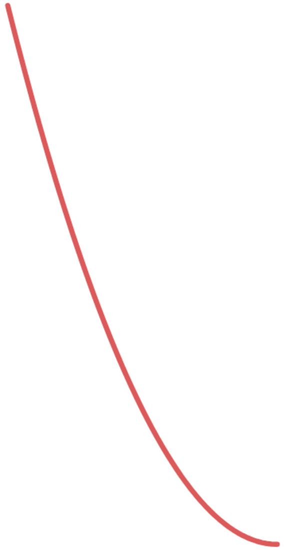
- If f is continuous on the interval I and $f''(x) > 0$ for all interior points x of the interval I, then f is convex on I.



If f is continuous on the interval I and $f''(x) < 0$ for all interior points x of the interval I, then f is concave on I.







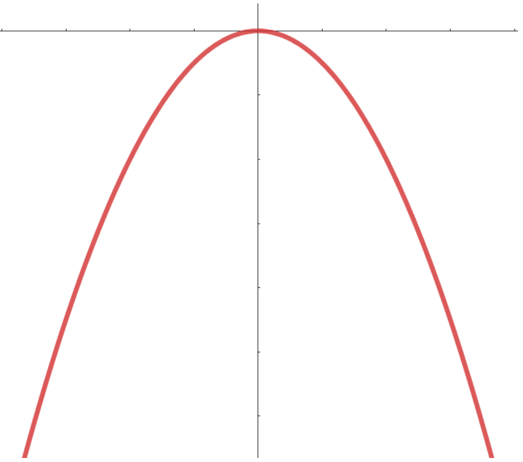
$$f''(x) > 0$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

> 0

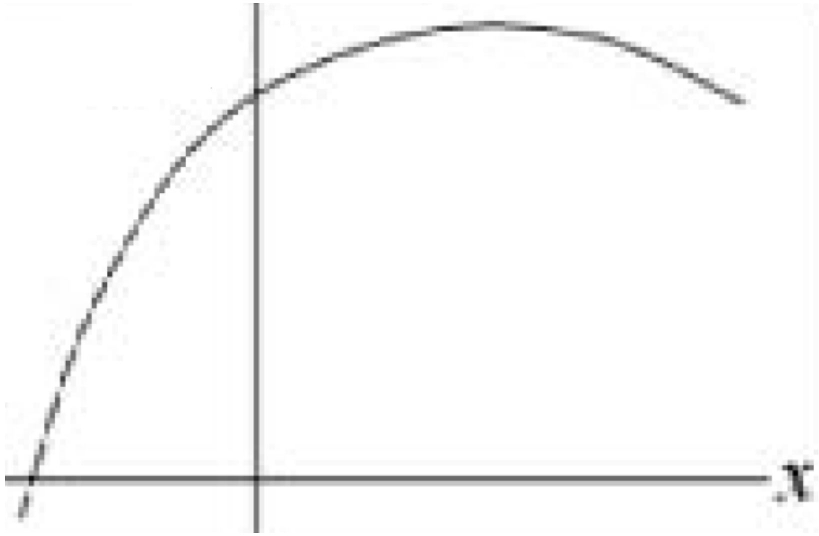
$$f''(x) \approx \frac{f'(x_2) - f'(x_1)}{x_2 - x_1} > 0$$

> 0

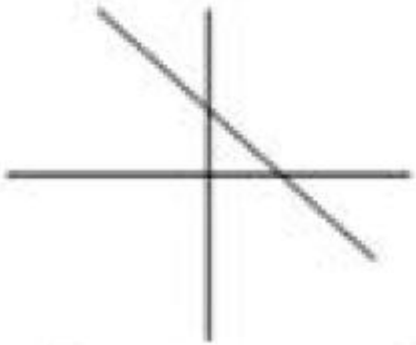


$$f''(x) < 0$$

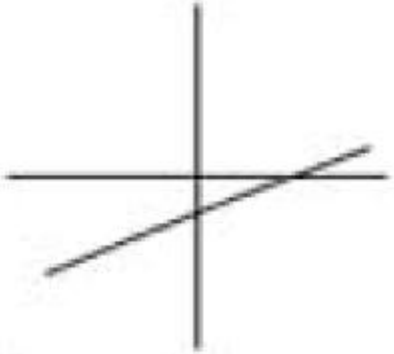
The graph of the function f is shown below. Which option from (a) to (e) could represent the graph of f'?



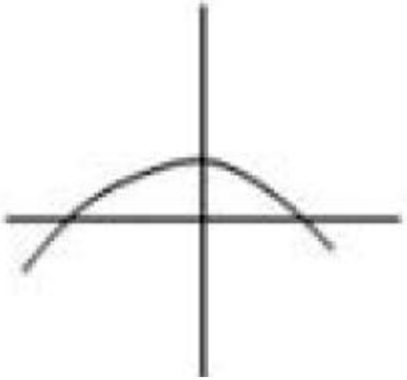
(a)



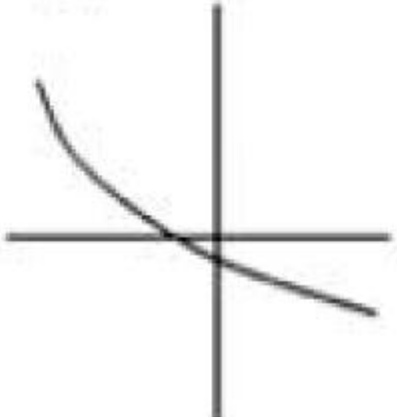
(b)



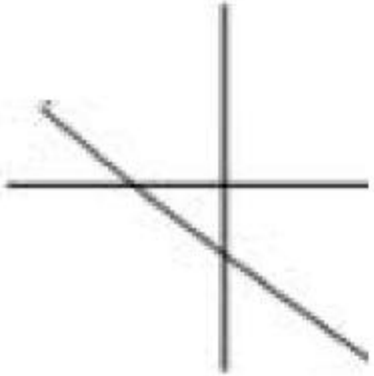
(c)



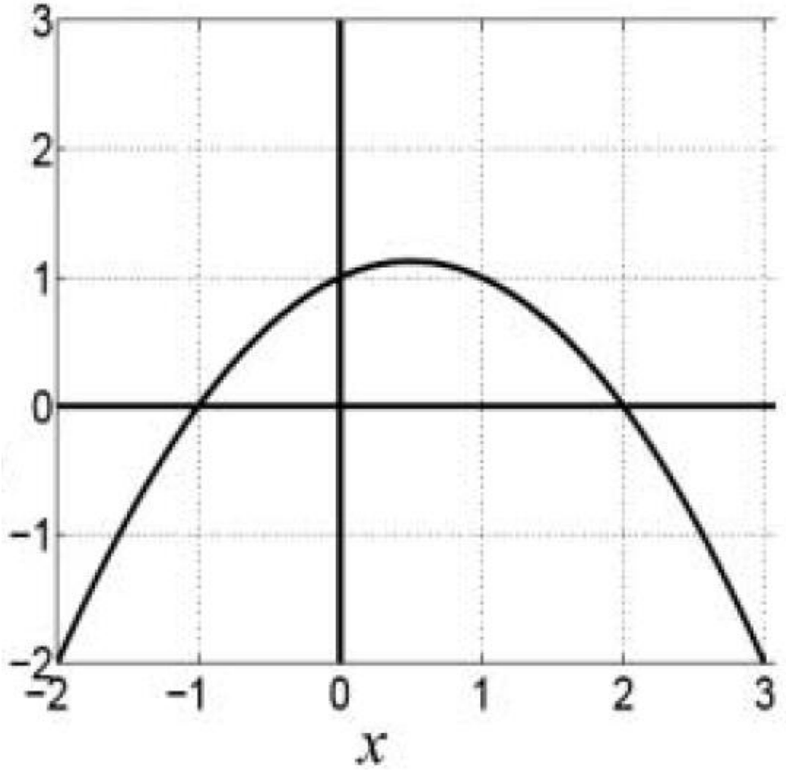
(d)



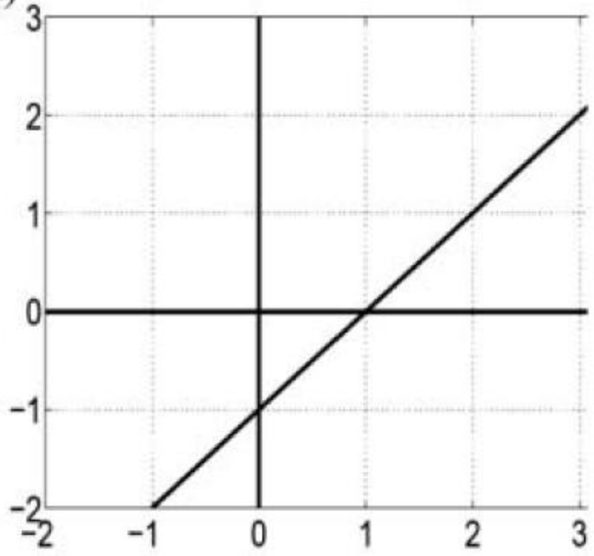
(e)



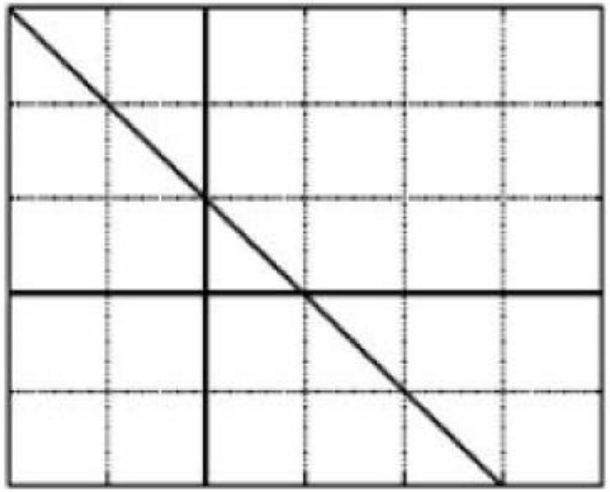
Which option from (a) to (e) could represent the graph of f?



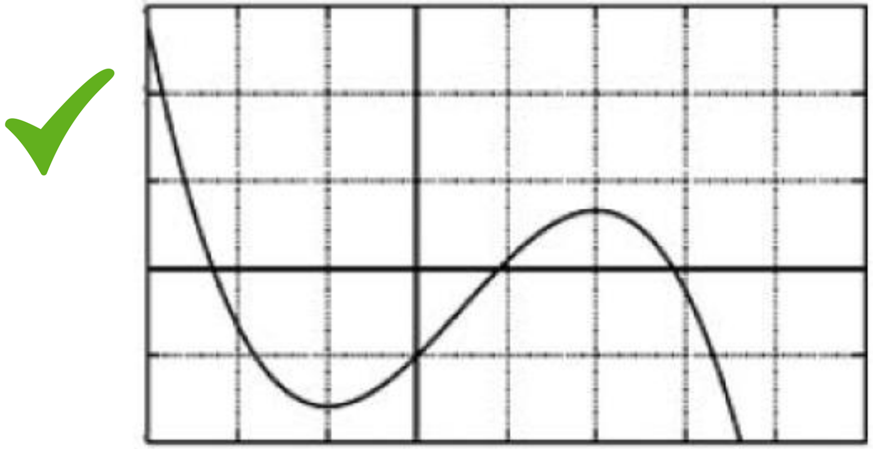
(a)



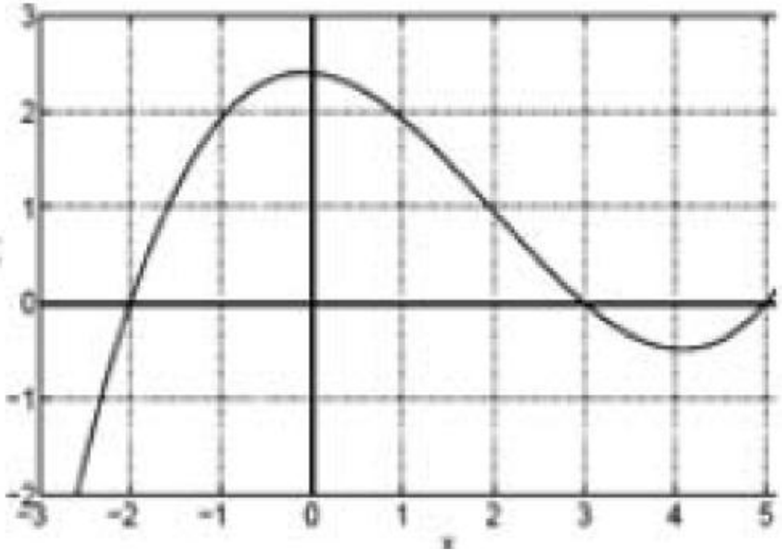
(b)



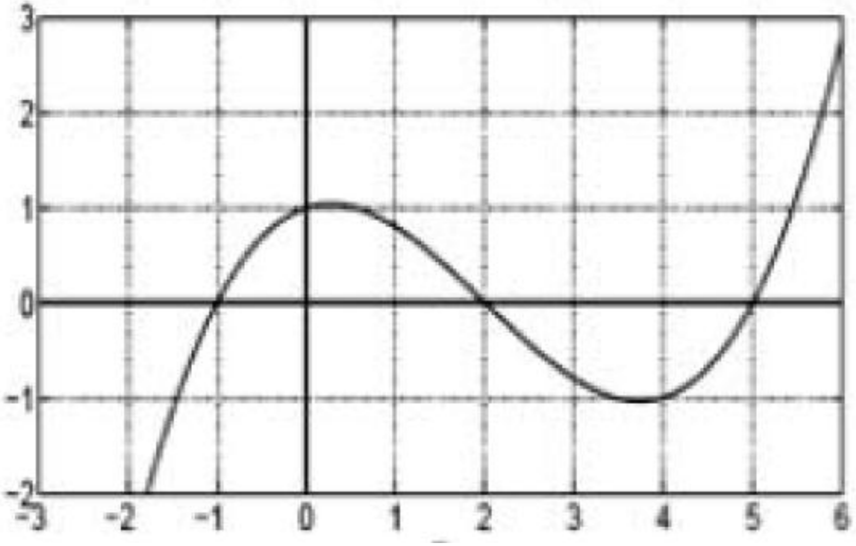
(c)



d)



e)



Homework. Based on the given graph of the function f, determine:

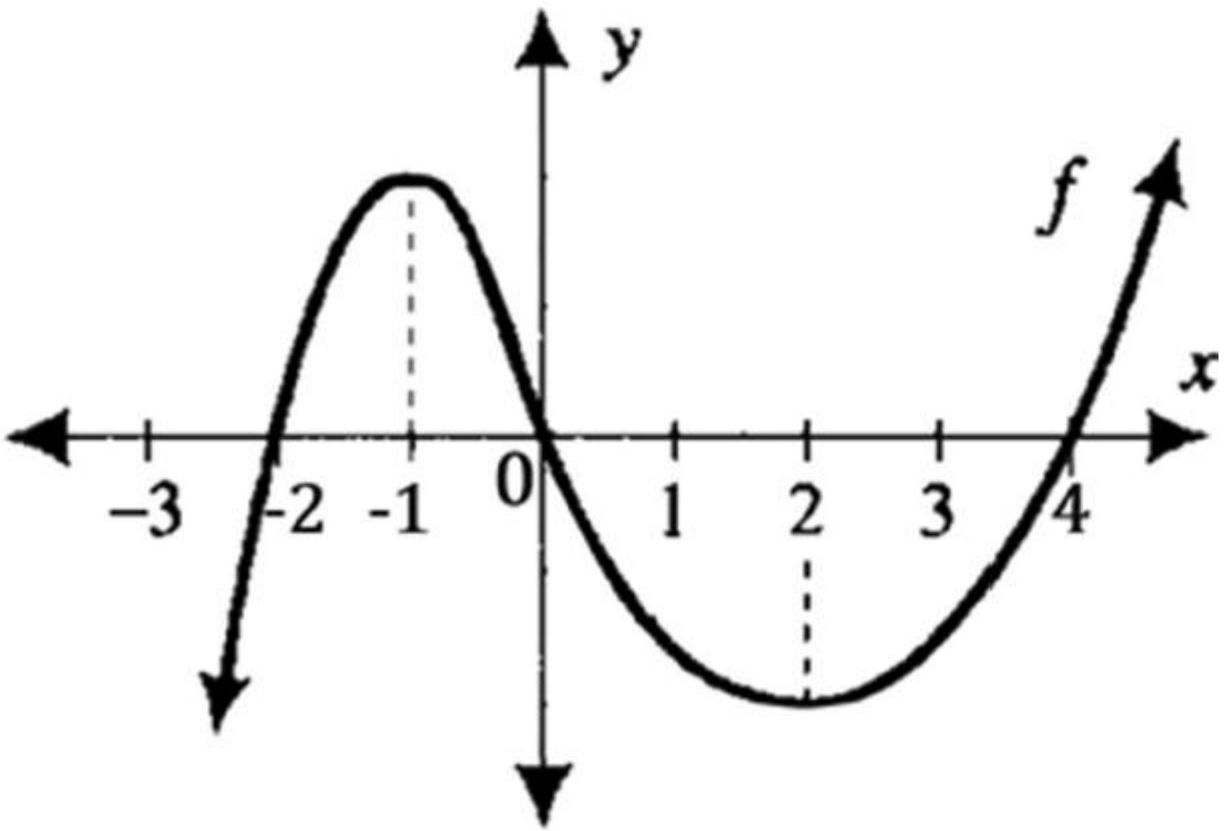
a) The intervals on which f is increasing and decreasing.

b) The point at which the function f has a local maximum and a local minimum.

c) The coordinates of the inflection points of f.

d) The intervals on which the function f is concave and convex.

e) Draw a possible graph of the derivative f'.



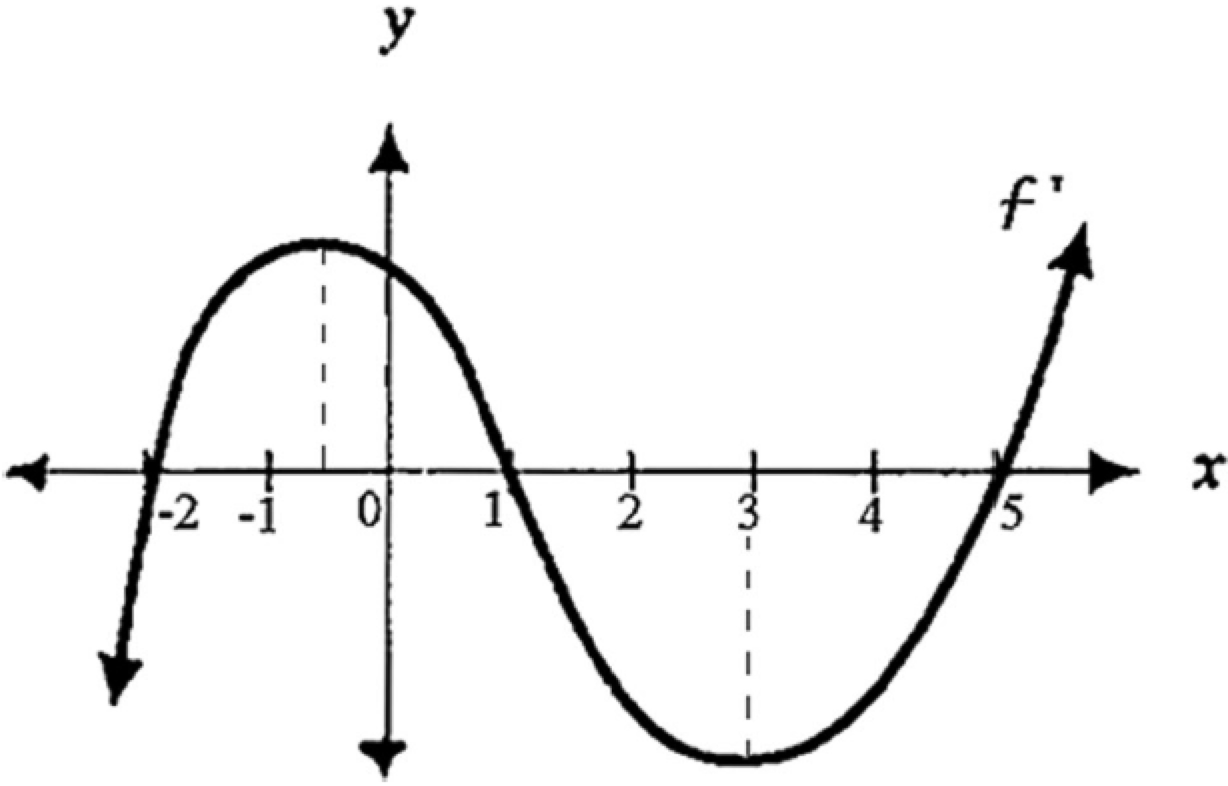
Homework. Given the graph of the derivative f', sketch a possible graph of the function f. Explain your reasoning and also determine:

a) The intervals where f is increasing or decreasing.

b) The maximum or minimum values of f.

c) The inflection points.

d) The intervals where f is convex or concave.



Sketch the graph of a function f that satisfies the following conditions:

The function f is continuous

$$f(0) = 2,\ f'(-2) = f'(3) = 0, \text{ a } \lim_{x \to 0} f'(x) = \infty$$

$$f'(x) > 0 \text{ when } -4 < x < -2, \text{ and when } -2 < x < 3,$$

$$f'(x) < 0 \text{ when } x < -4, \text{ and when } x > 3,$$

$$f''(x) < 0 \text{ when } x < -4, \text{ when } -4 < x < -2, \text{ and when } 0 < x < 5,$$

$$f''(x) > 0 \text{ when } -2 < x < 0, \text{ and when } x > 5,$$

$$\lim_{x \to -\infty} f(x) = \infty \text{ and } \lim_{x \to \infty} f(x) = -2$$

References

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Thank you for your attention!

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