Charles University

Faculty of Mathematics and Physics;

Departemnet of Mathematics Education

Developing conceptual knowledge in school mathematics

Lesson #3

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Interpretation of Exponential and Logarithmic Functions in Contextual Situations

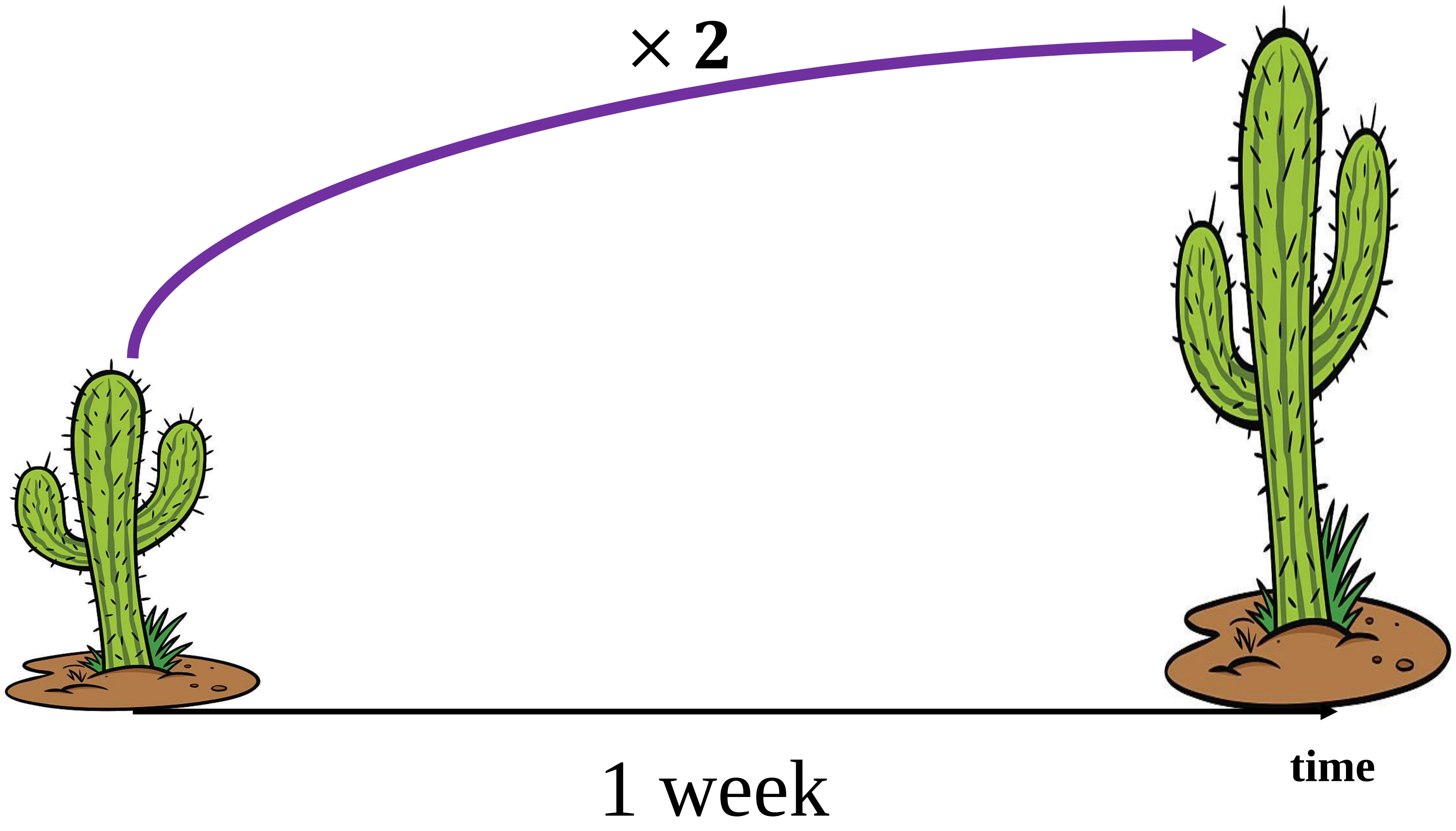
Some real-world situations that we experience in everyday life, or some pseudorealistic situations.

Goal: How can we help students and teachers understand the concepts of the exponent and the logarithm and their rules in real-life situations?

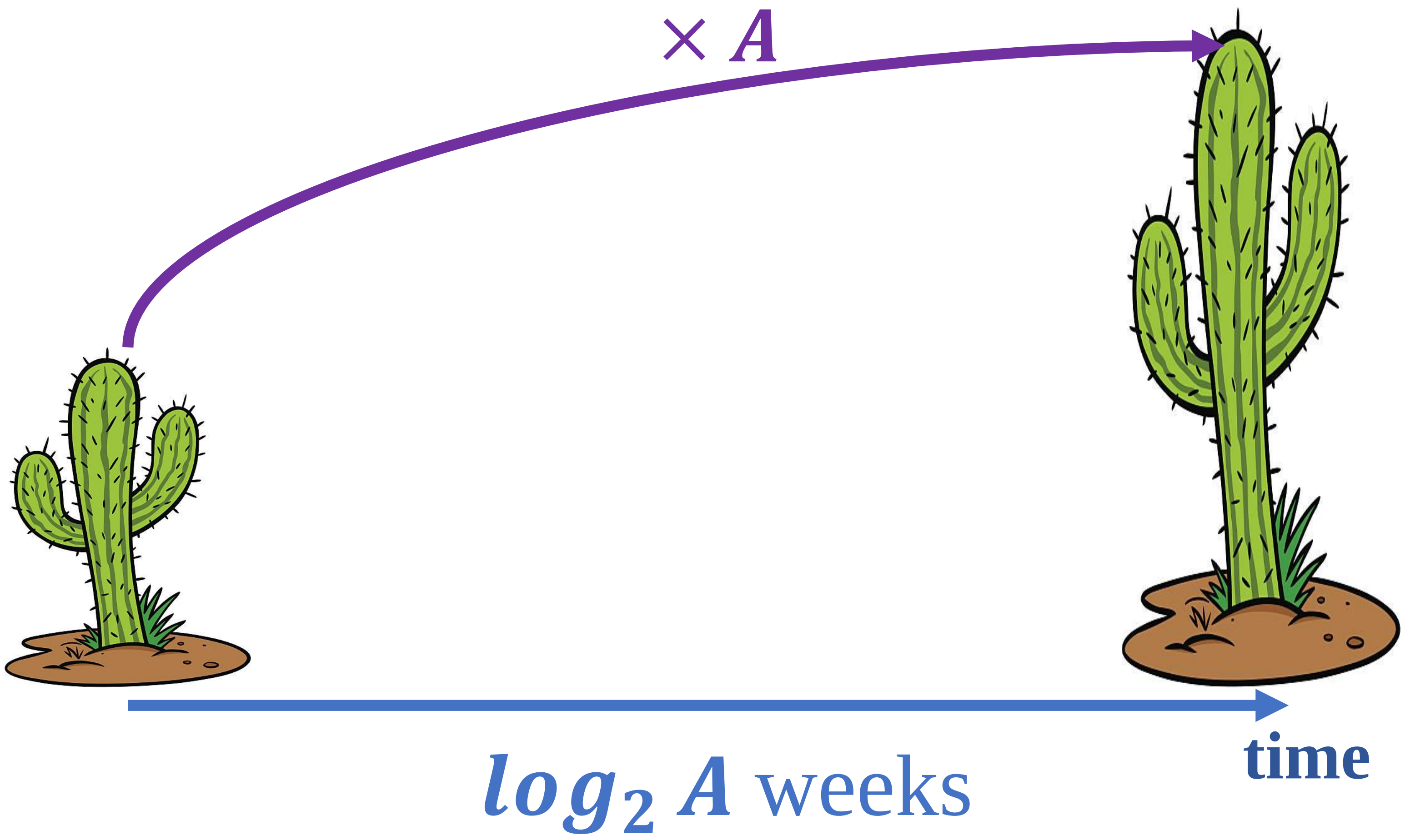
$$\mathbf{2^{\frac{1}{7}}} \qquad \log_2 \mathbf{3}$$

$$\log_2 \mathbf{7} + \log_2 \mathbf{4} = \log_2 \mathbf{28}$$

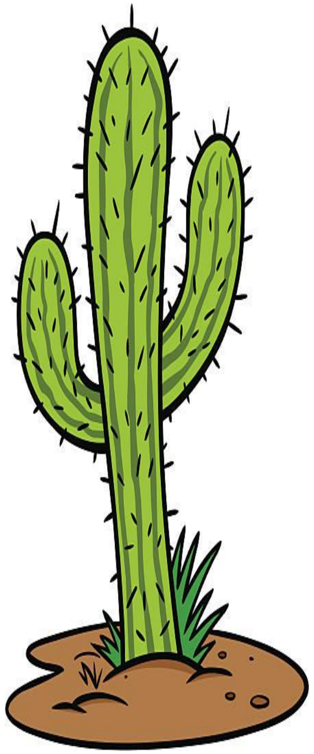
We use a story about a cactus.

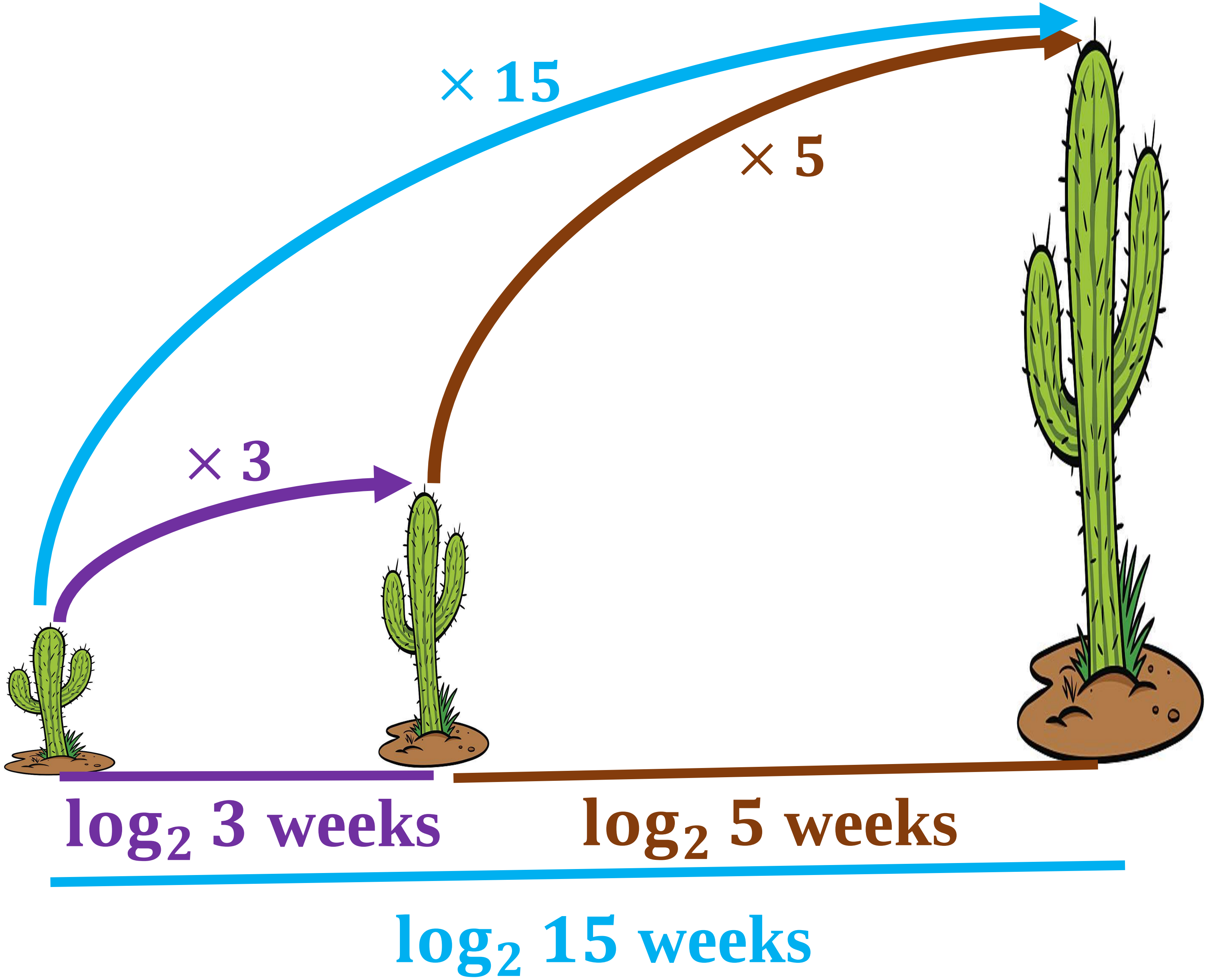


$\boldsymbol{log_2 A}$ expresses the number of weeks the cactus needs to reach **$\boldsymbol{A}$-times** its height.



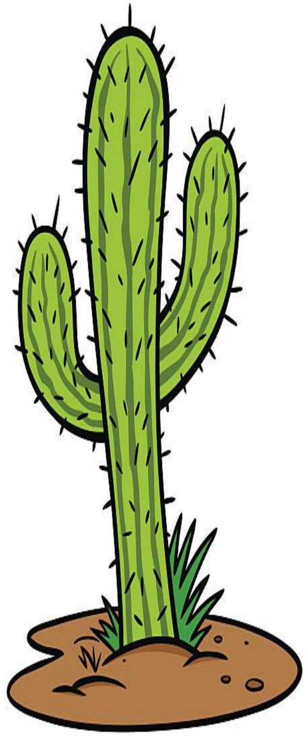
$$\log_2 3 + \log_2 5 = \log_2 15$$



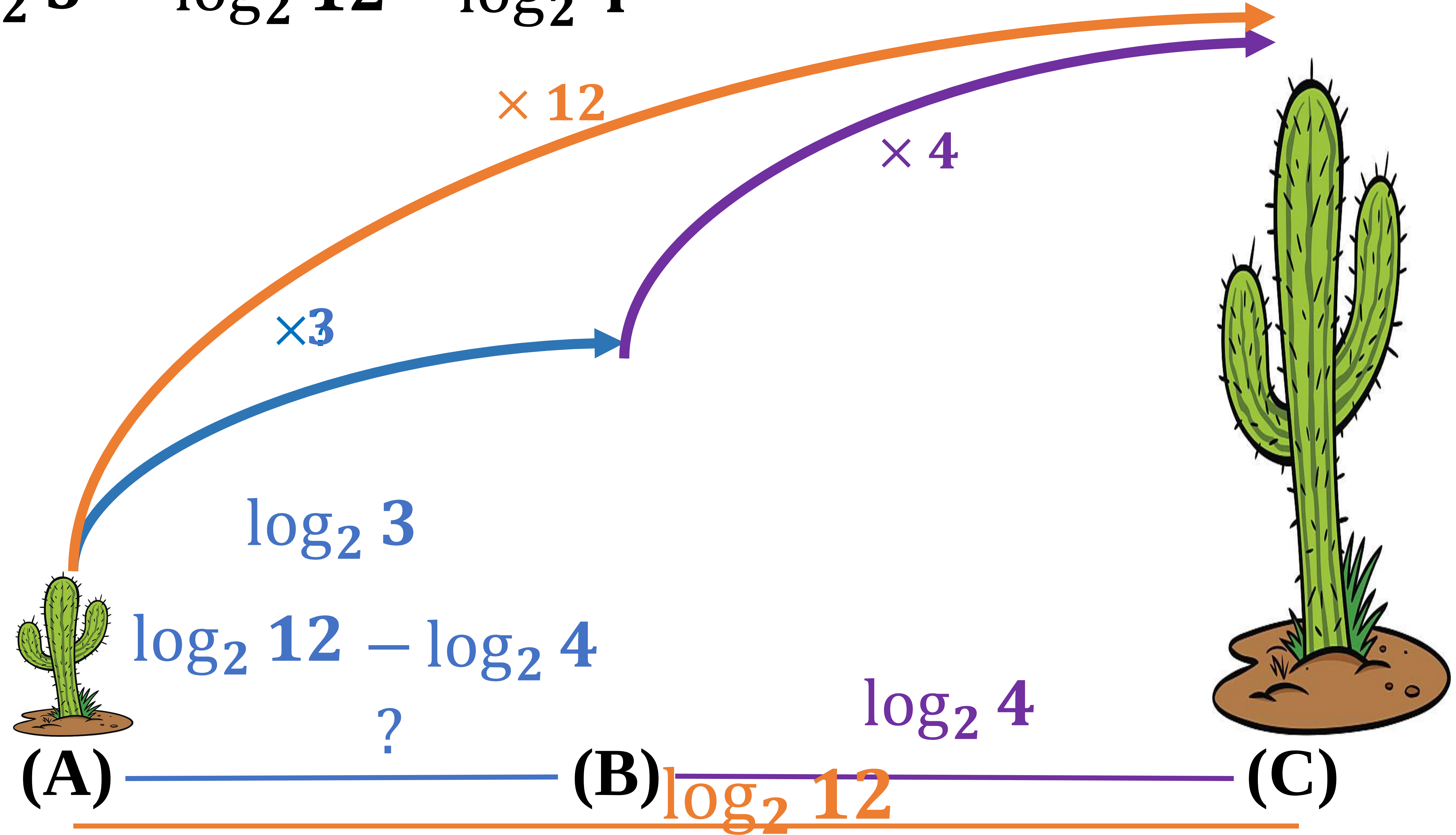


$$\log_2 3 + \log_2 5 = \log_2 15$$

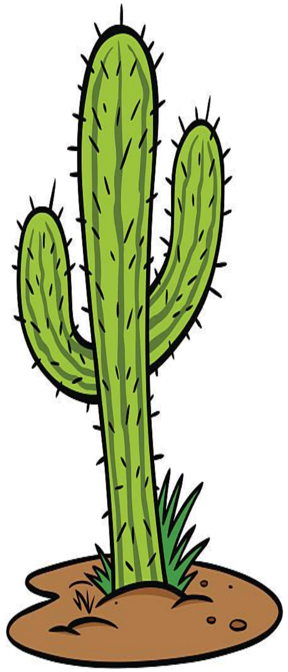
$$\log_2 12 - \log_2 4 = \log_2 3$$

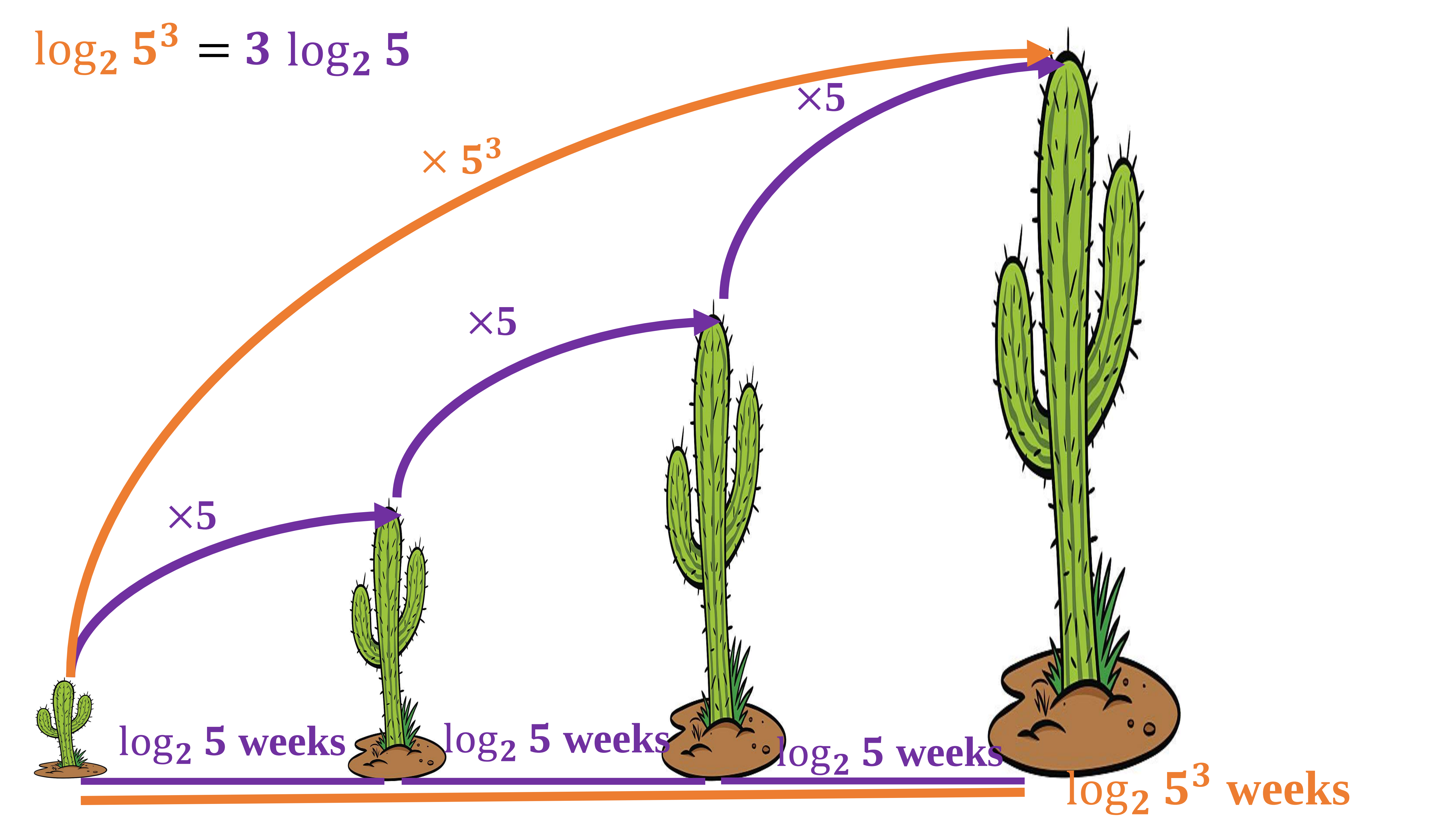


$$\log_2 \mathbf{3} = \log_2 \mathbf{12} - \log_2 \mathbf{4}$$



$$\log_{\mathbf{2}} \mathbf{5}^{3} = \mathbf{3} \log_{\mathbf{2}} \mathbf{5}$$





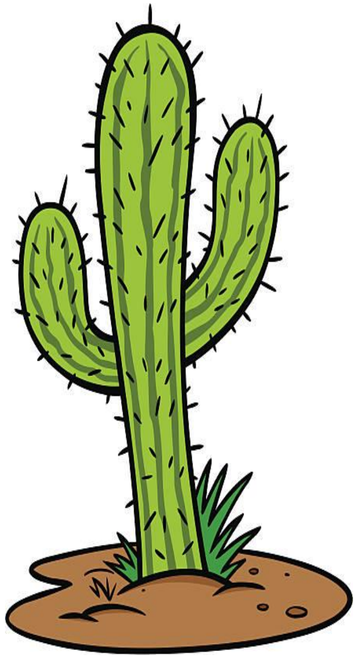
$\log_2 5^3 = 3 \log_2 5$
$\times 5^3$
$\times 5$
$\times 5$
$\times 5$
$\log_2 5$ weeks
$\log_2 5$ weeks
$\log_2 5$ weeks
$\log_2 5^3$ weeks

Other logarithmic rules can also be interpreted using the cactus story.

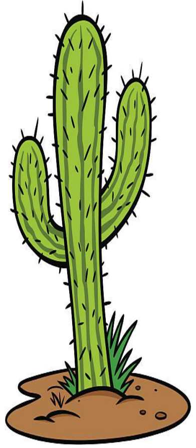
$$\log_2 2^A = A$$

$$2^{\log_2 A} = A$$

$$\frac{\log_2 A}{\log_2 B} = \log_B A$$



We have a cactus whose height doubles every week. Fill in the blanks.



a) The number of weeks the cactus needs to reach 2^3-times its height is … 3

b) The number of weeks the cactus needs to reach 2^4-times its height is … 4

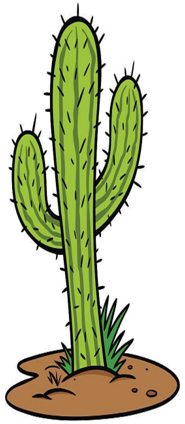
c) The number of weeks the cactus needs to reach 2^{100}-times its height is … 100

d) The number of weeks the cactus needs to reach 2^A-times its height is … A

e) Write the statement from part (d) as a logarithmic equation.

$$\log_2 \mathbf{2}^{\boldsymbol{A}} = \boldsymbol{A}$$

We have a cactus whose height doubles every week. Fill in the blanks.



a) $2^{\text{the number of weeks the cactus needs to reach } 8-\text{times its height}}$ is 8…

b) $2^{\text{the number of weeks the cactus needs to reach } 9-\text{times its height}}$ is 9…

c) $2^{\text{the number of weeks the cactus needs to reach } 10-\text{times its height}}$ is 10…

d) $2^{\text{the number of weeks the cactus needs to reach } 100-\text{times its height}}$ is 100…

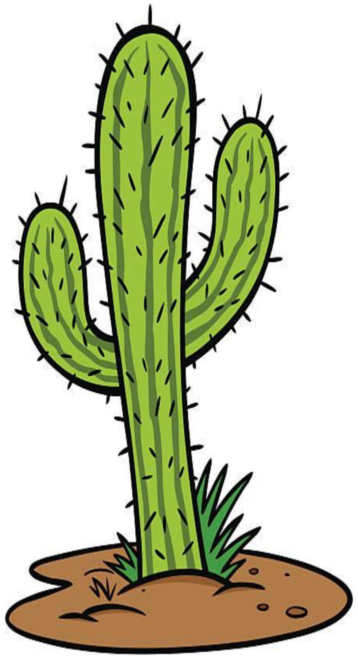
e) $2^{\text{the number of weeks the cactus needs to reach } A-\text{times its height}}$ is A…

f) Write the statement from part (e) as a logarithmic equation.

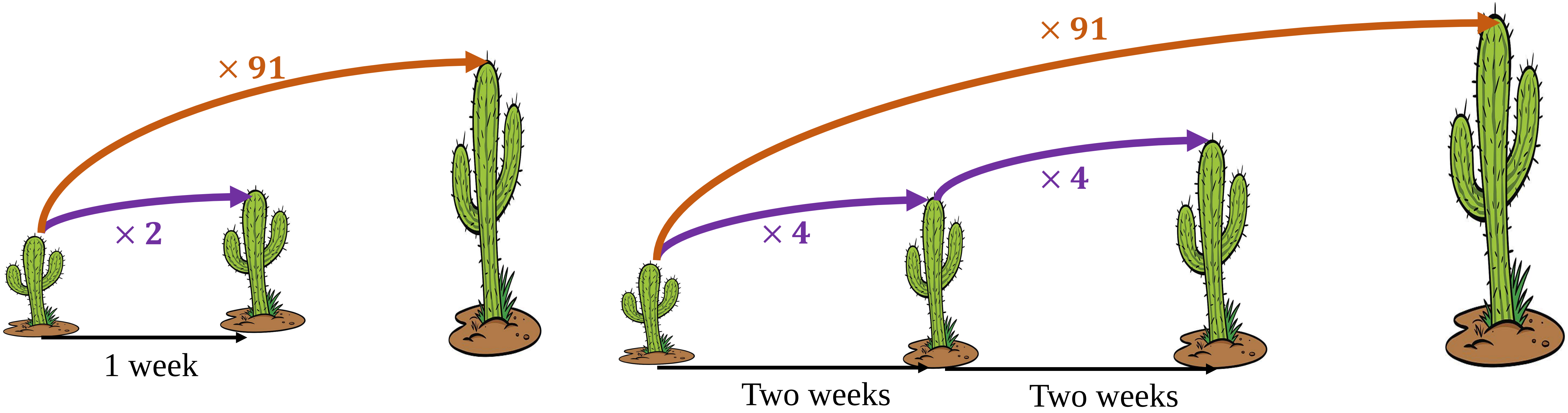
$$\mathbf{2^{\log_2 A} = A}$$

Use the cactus story and explain why this logarithmic statement is true.

$$\frac{\log_2 \boldsymbol{A}}{\log_2 \boldsymbol{B}} = \log_{\boldsymbol{B}} \boldsymbol{A}$$



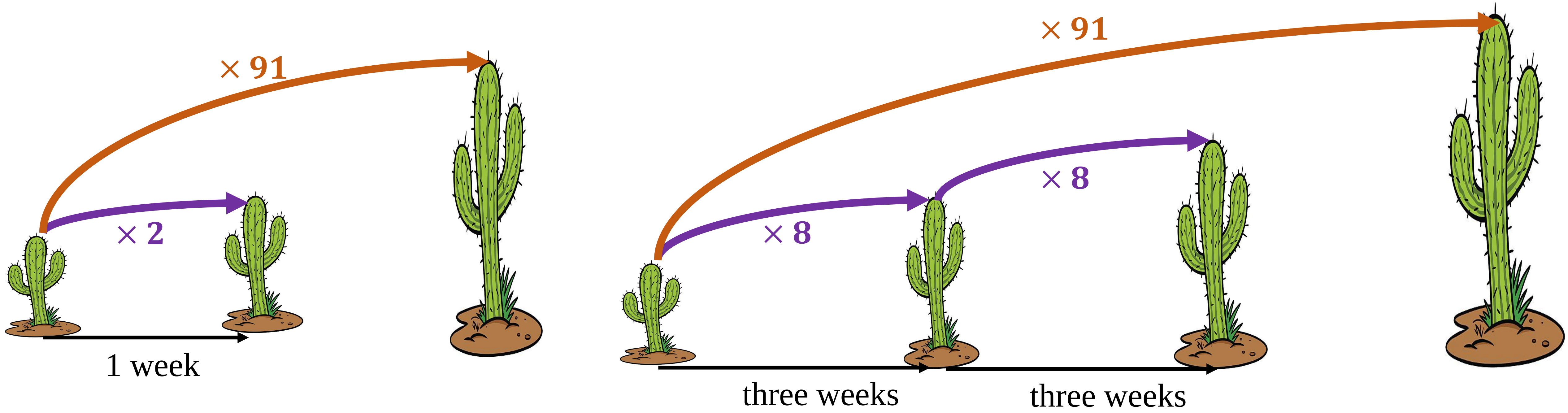
$$\frac{\log_2 \mathbf{15}}{\log_2 \mathbf{5}} = \frac{\log_4 \mathbf{15}}{\log_4 \mathbf{5}} = \frac{\log_3 \mathbf{15}}{\log_3 \mathbf{5}} = \log_5 \mathbf{15}$$



$\log_2 91$ represents the number of weeks the cactus needs to reach 91 times its height.

two-week periods

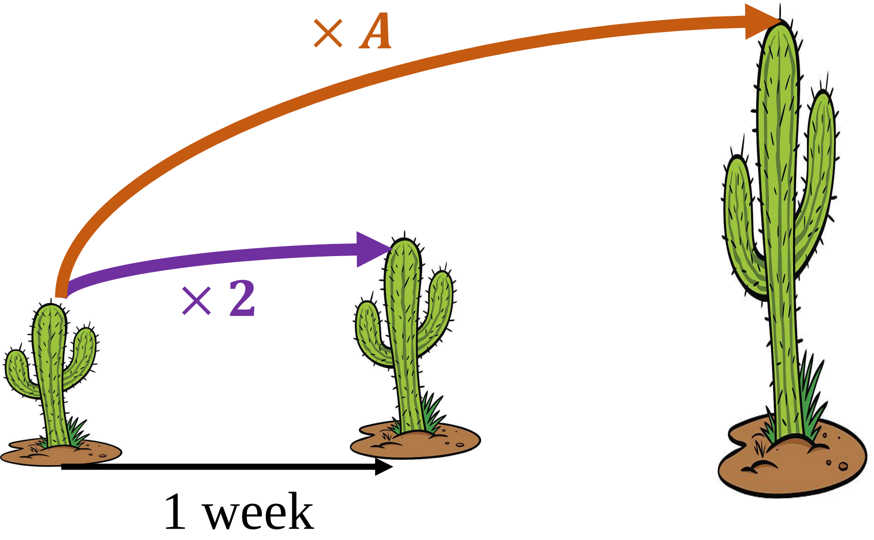
$\log_4 91$ represents the number of ………. the cactus needs to reach 91 times its original height.

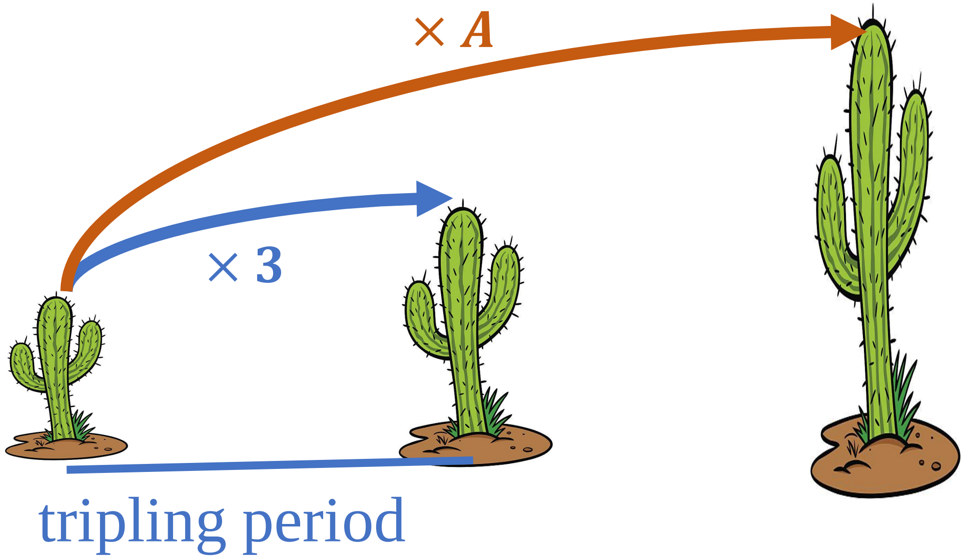


$\mathbf{\log_2 91}$ represents the number of weeks the cactus needs to reach 91 times its height.

three-week periods

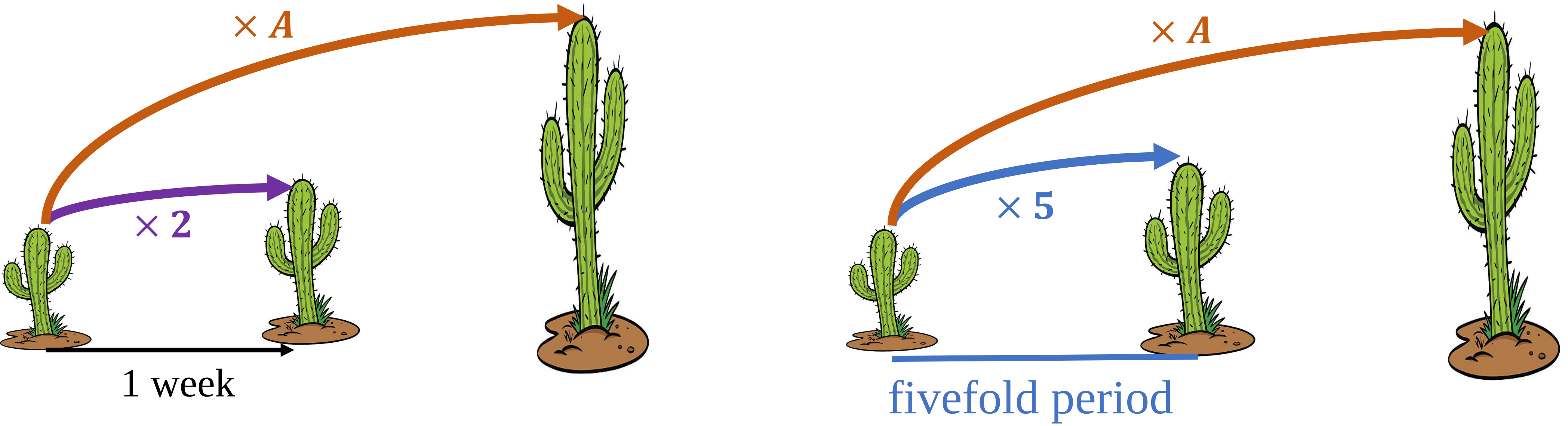
$\mathbf{\log_8 91}$ represents the number of ………. the cactus needs to reach 91 times its height.





= is the amount of time it takes for the height of the cactus to triple.

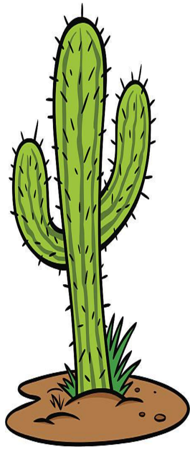
$\log_3 A = ?$ How many **tripling periods** does the cactus need to reach *A times* its height?



= is the amount of time it takes for the height of the cactus to increase fivefold.

$\log_5 A = ?$ How many **fivefold periods** does the cactus need to reach A *times* its original height?

We have a cactus whose height doubles every week. Answer the following questions (your answer should be expressed in logarithmic form).



How many doubling periods does the cactus need to reach nine times its original height? $\log_2 9$

How many quadrupling periods does the cactus need to reach five times its original height? $\log_4 5$

How many sixfold periods does the cactus need to reach 200 times its original height? $\log_6 200$

How many tripling periods does the cactus need to reach eight times its original height? $\log_3 8$

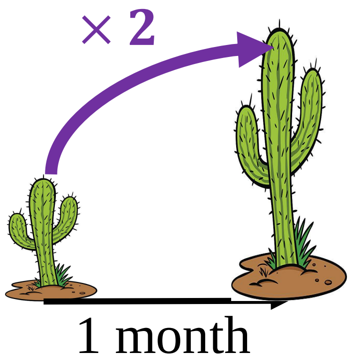
$$\log_{y^a} x = \frac{1}{a} \log_y x$$

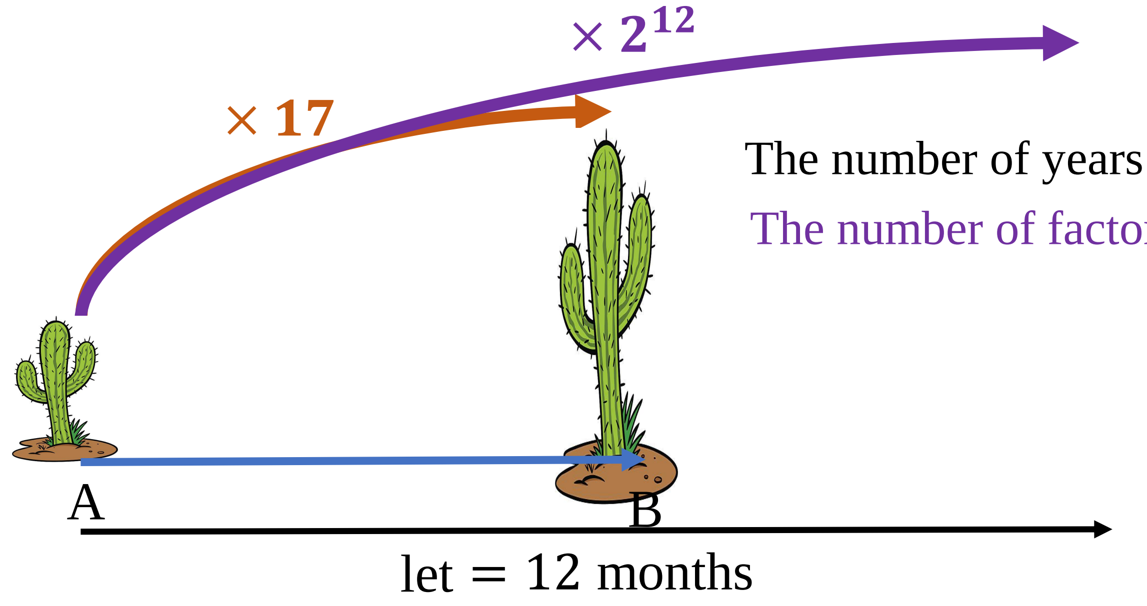
$$\log_{2^{12}} 17 = \frac{1}{12} \log_2 17$$

Homework: We have a cactus whose height doubles every month. Using this cactus, design a simple word problem that demonstrates that the equality $\log_{2^{12}} 17 = \frac{1}{12}\log_2 17$ holds.

Hint: You must answer your word problem in two different ways. In the first method, you must obtain $\log_{2^{12}} 17$ as the answer. In the second method, you must obtain $\frac{1}{12}\log_2 17$. Since the word problem has only one correct answer, you may conclude that the equality $\log_{2^{12}} 17 = \frac{1}{12}\log_2 17$ holds.

We have a cactus whose height doubles every month. Determine the number of years the cactus needs to become seventeen times its original height.



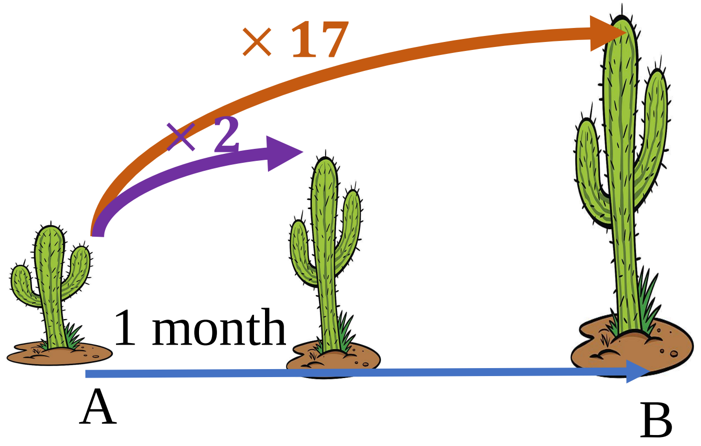


The number of years the cactus needs to reach ***17 times*** its height.

The number of factors $\mathbf{2^{12}}$ that are contained in the number 17

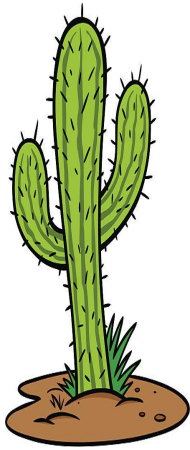
$$\log_{2^{12}} 17 \text{ years}$$

$$\log_{2^{12}} 17 = \frac{1}{12}\log_2 17$$



$$\log_2 17 \text{ months} \longrightarrow \frac{1}{12}\log_2 17 \text{ years}$$

We have a cactus whose height doubles every week. Correct the following equality by multiplying or dividing one side by a number.



$\frac{1}{2}\times$ The number of doubling periods the cactus needs to reach *eight times* its original height

=

The number of quadrupling periods the cactus needs to reach *eight times* its original height.

We have a cactus whose height doubles every week. Simplify each fraction/ratio.

a)

$$\frac{\text{The number of doubling periods the cactus needs to reach } 2^k \textit{ times} \text{ its original height}}{\text{The number of doubling periods the cactus needs to reach } \textit{twice} \text{ its original height}} = k$$

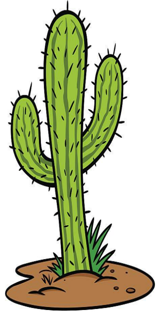
b)

$$\frac{\text{The number of quadrupling periods the cactus needs to reach } 2^k \textit{ times} \text{ its original height}}{\text{The number of quadrupling periods the cactus needs to reach } \textit{twice} \text{ its original height}} = k$$

c)

$$\frac{\text{The number of fivefold periods the cactus needs to reach } 2^k \text{ times its original height}}{\text{The number of fivefold periods the cactus needs to reach twice its original height.}} = k$$

Use the cactus story to explain why this logarithmic theorem is true.



$$\frac{\log_2 \mathbf{15}}{\log_2 \mathbf{5}} = \frac{\log_4 \mathbf{15}}{\log_4 \mathbf{5}} = \log_5 \mathbf{15}$$

$$\frac{\log_2 \mathbf{15}}{\log_2 \mathbf{5}} = \frac{\textbf{The number of doubling periods}\text{ the cactus needs to reach }\mathbf{15}\ \textbf{\textit{times}}\text{ its height}}{\textbf{The number of doubling periods}\text{ the cactus needs to reach }\mathbf{5}\ \textbf{\textit{times}}\text{ its height}}$$

$$\frac{\log_4 \mathbf{15}}{\log_4 \mathbf{5}} = \frac{\textbf{The number of quadrupling periods}\text{ the cactus needs to reach }\mathbf{15}\ \textbf{\textit{times}}\text{ its height}}{\textbf{The number of quadrupling periods}\text{ the cactus needs to reach }\mathbf{5}\ \textbf{\textit{times}}\text{ its height}}$$

$$= \frac{\textbf{The number of fivefold periods}\text{ the cactus needs to reach }\textbf{15 times}\text{ its original height}}{\textbf{The number of fivefold periods}\text{ the cactus needs to reach }\textbf{5 times}\text{ its original height}}$$

$$\frac{\log_2 \mathbf{15}}{\log_2 \mathbf{5}} = \frac{\log_4 \mathbf{15}}{\log_4 \mathbf{5}} = \frac{\log_5 \mathbf{15}}{\log_5 \mathbf{5}}$$

$$\log_{10} x + \log_{10} 3 = 2\log_{10} 4 - \log_{10} 2$$

$$log_{10} 3x = log_{10} 16 - log_{10} 2$$

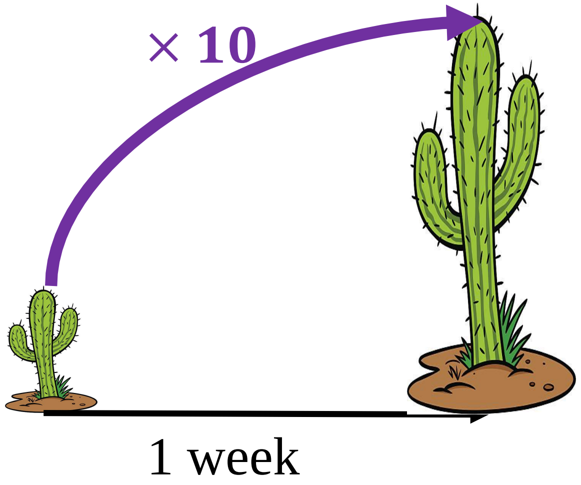
$$log_{10} 3x = log_{10} 8$$

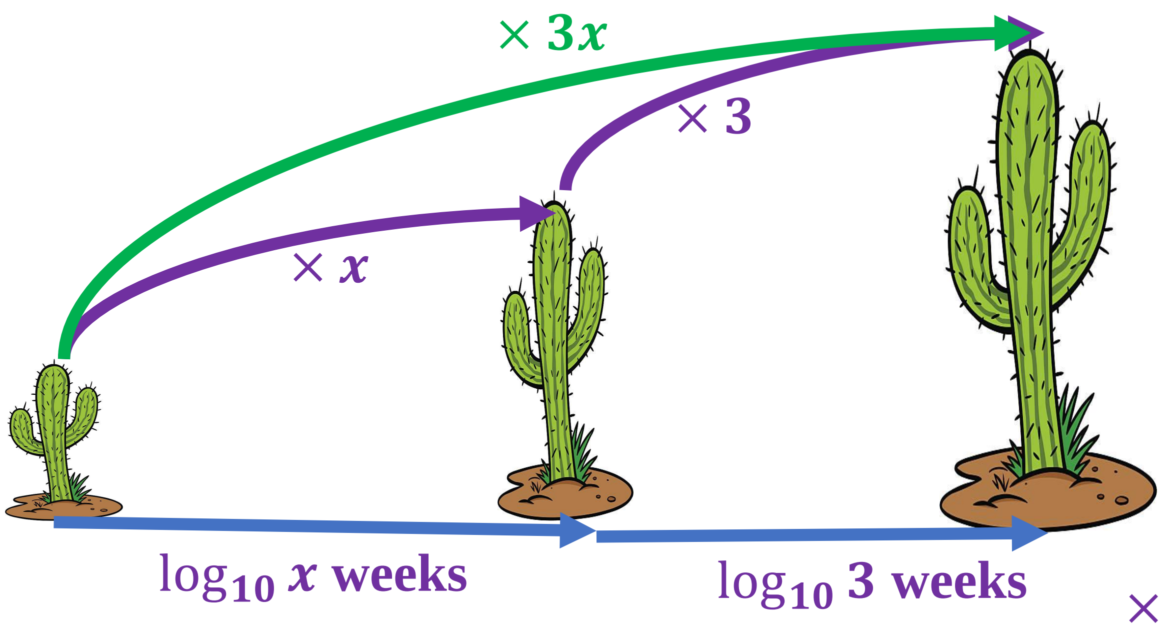
$$3x = 8$$

$$x = \frac{8}{3}$$

What does each step mean? What does the answer mean?

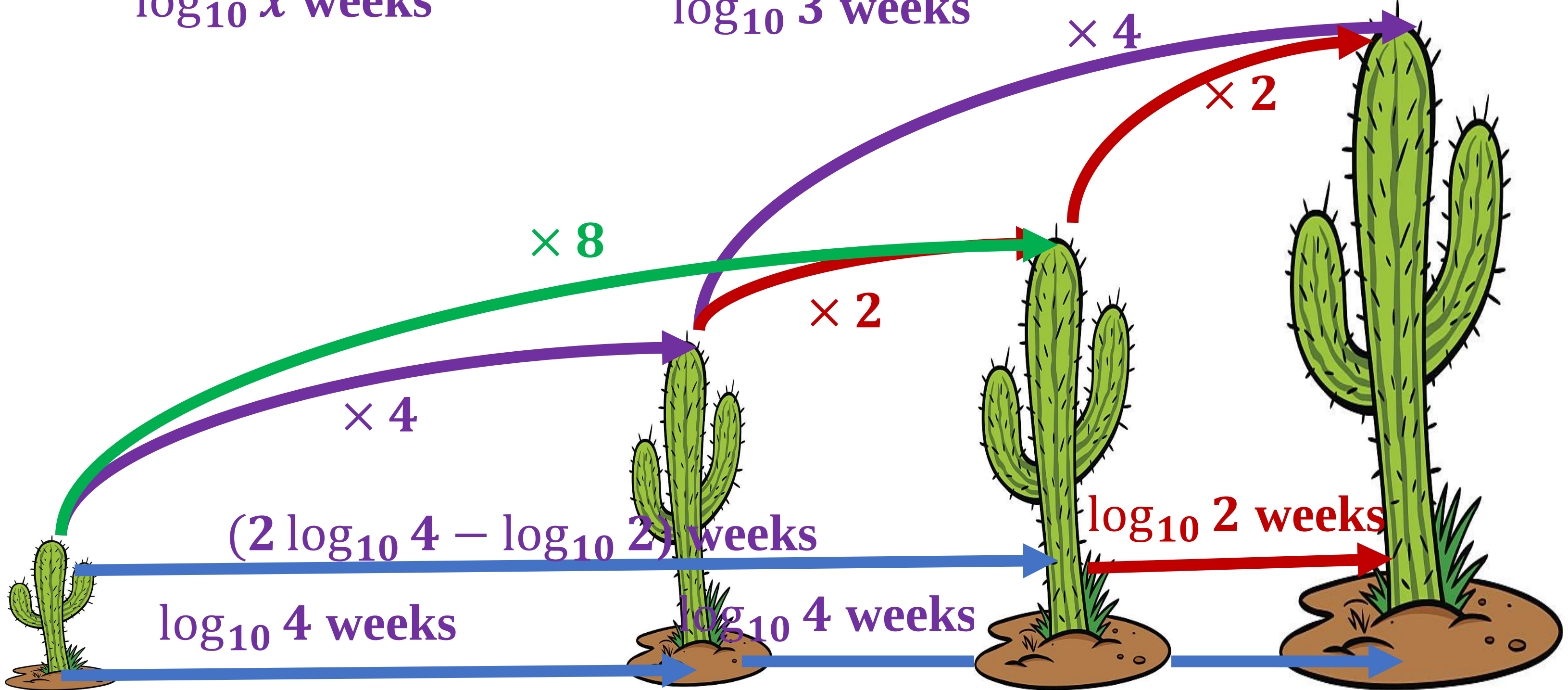
$$\log_{10} x + \log_{10} 3 = 2\log_{10} 4 - \log_{10} 2$$



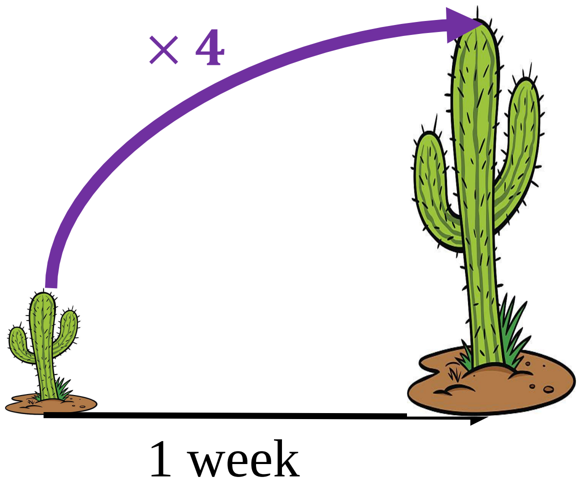


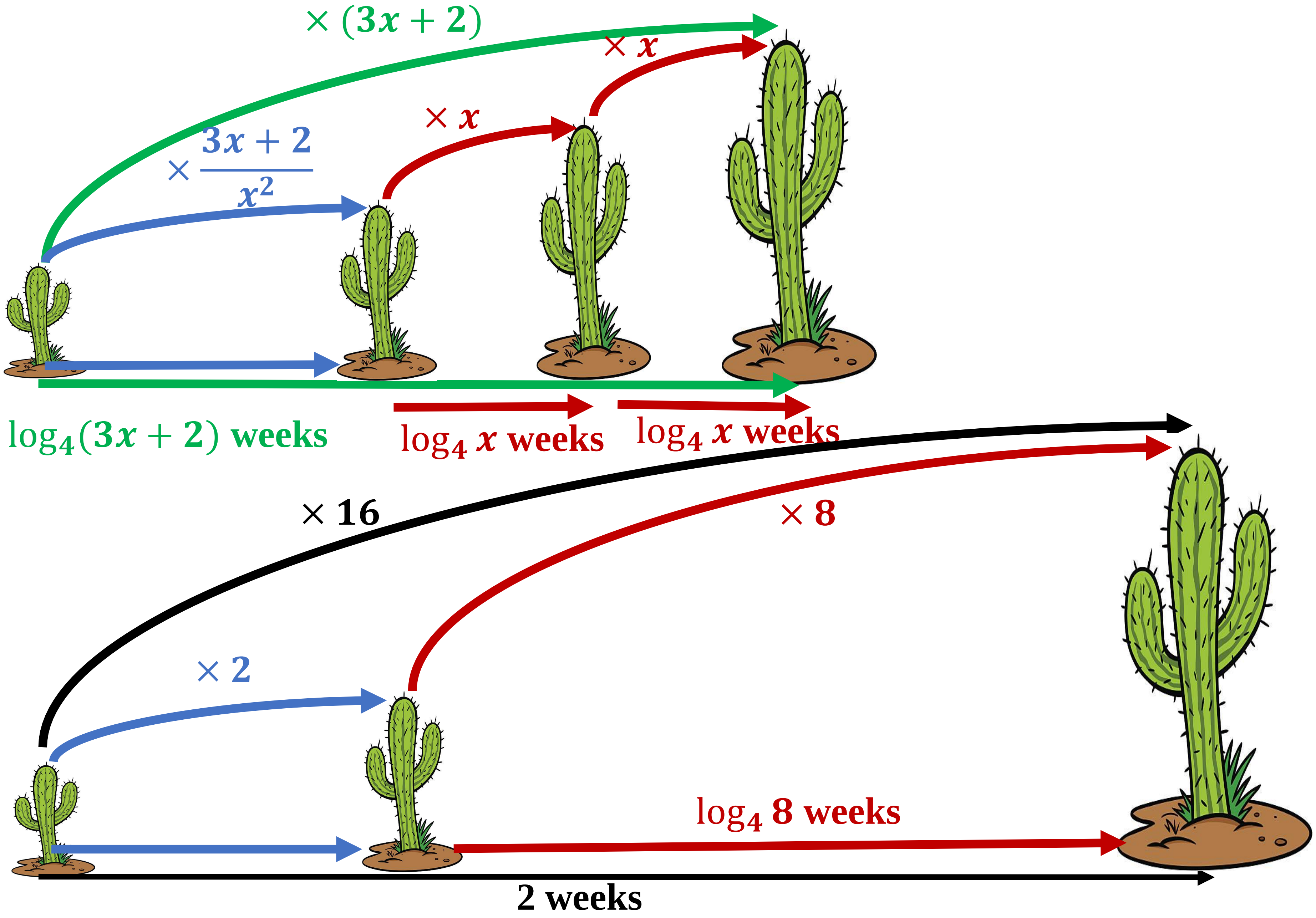
$$3x = 8$$

$$x = \frac{8}{3}$$



$$\log_4(3x+2) - 2\log_4 x = 2 - \log_4 8$$





$$\frac{3x+2}{x^2} = 2$$

$$\boxed{x = 2}$$

$$x = -\frac{1}{2} \quad \times$$

(Homework) Explain whether the topics covered in these three sessions changed your perspective on logarithms. What new things did you learn?

References

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Děkuji za pozornost!

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