Charles University

Faculty of Mathematics and Physics;

Departemnet of Mathematics Education

Developing conceptual knowledge in school mathematics

Lesson #2

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The aim of this course:

- Conceptual understanding
- Secondary school level
- Problem-solving
- Homework (borji@karlin.mff.cuni.cz) [Tuesday 12:00]

Topics of the course:

- Exponential and logarithmic functions
- Trigonometric functions and their inverses
- Derivatives
- Limits of sequences
- Series and their convergence
- Combinatorics

Interpretation of Exponential and Logarithmic Functions in Contextual Situations

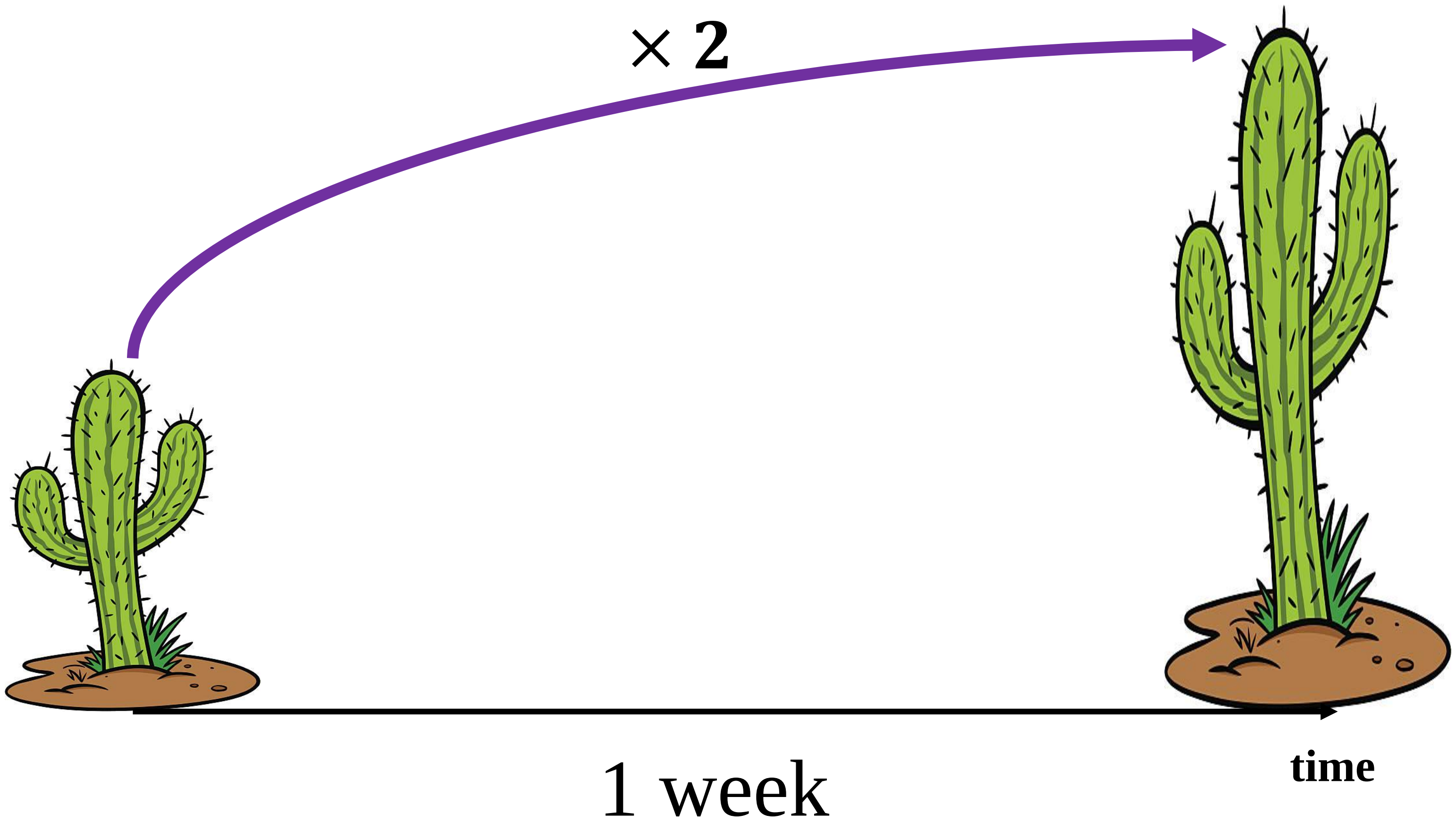
Some real-world situations that we experience in everyday life, or some pseudorealistic situations.

Goal: How can we help students and teachers understand the concepts of the exponent and the logarithm and their rules in real-life situations?

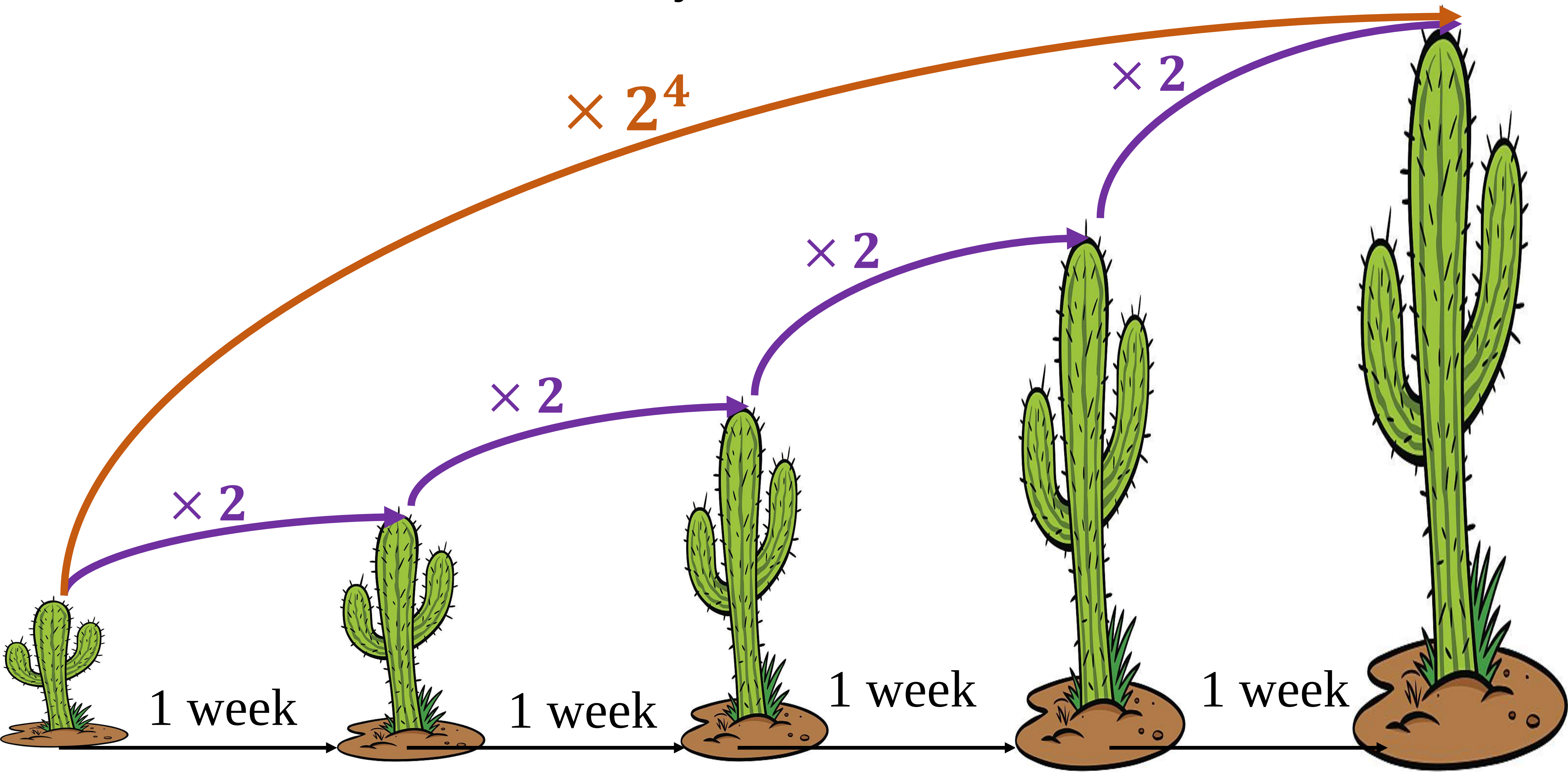
$$\mathbf{2^{\frac{1}{7}}} \qquad \log_2 \mathbf{3}$$

$$\log_2 \mathbf{7} + \log_2 \mathbf{4} = \log_2 \mathbf{28}$$

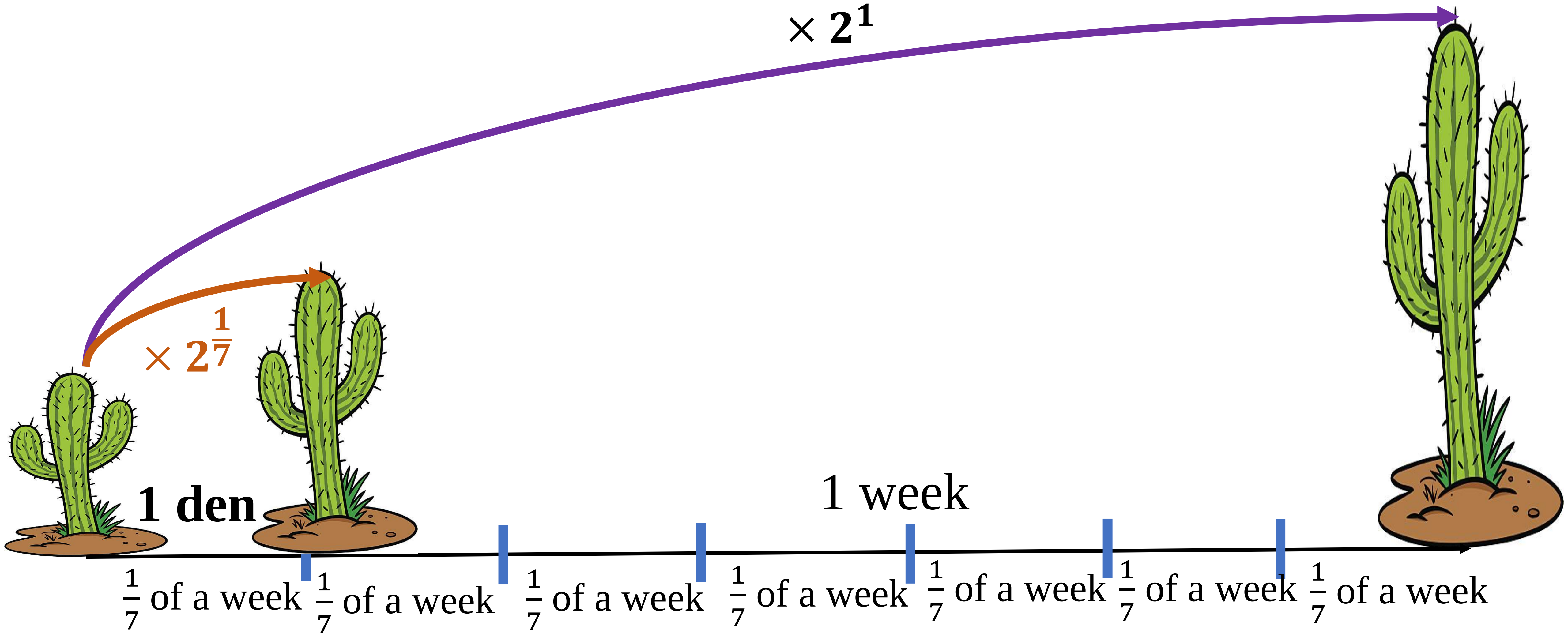
We use a story about a cactus.



What does $\mathbf{2}^4$ mean in this story?

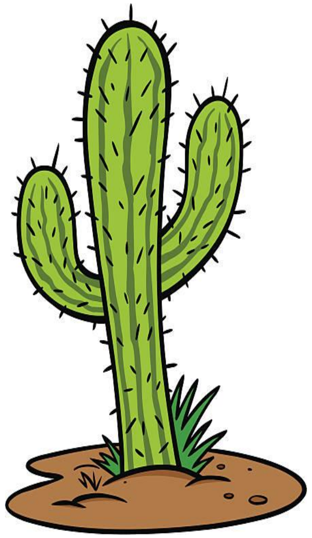


What does $2^{\frac{1}{7}}$ mean in this story?



What does $\mathbf{2^0}$ mean in this story?

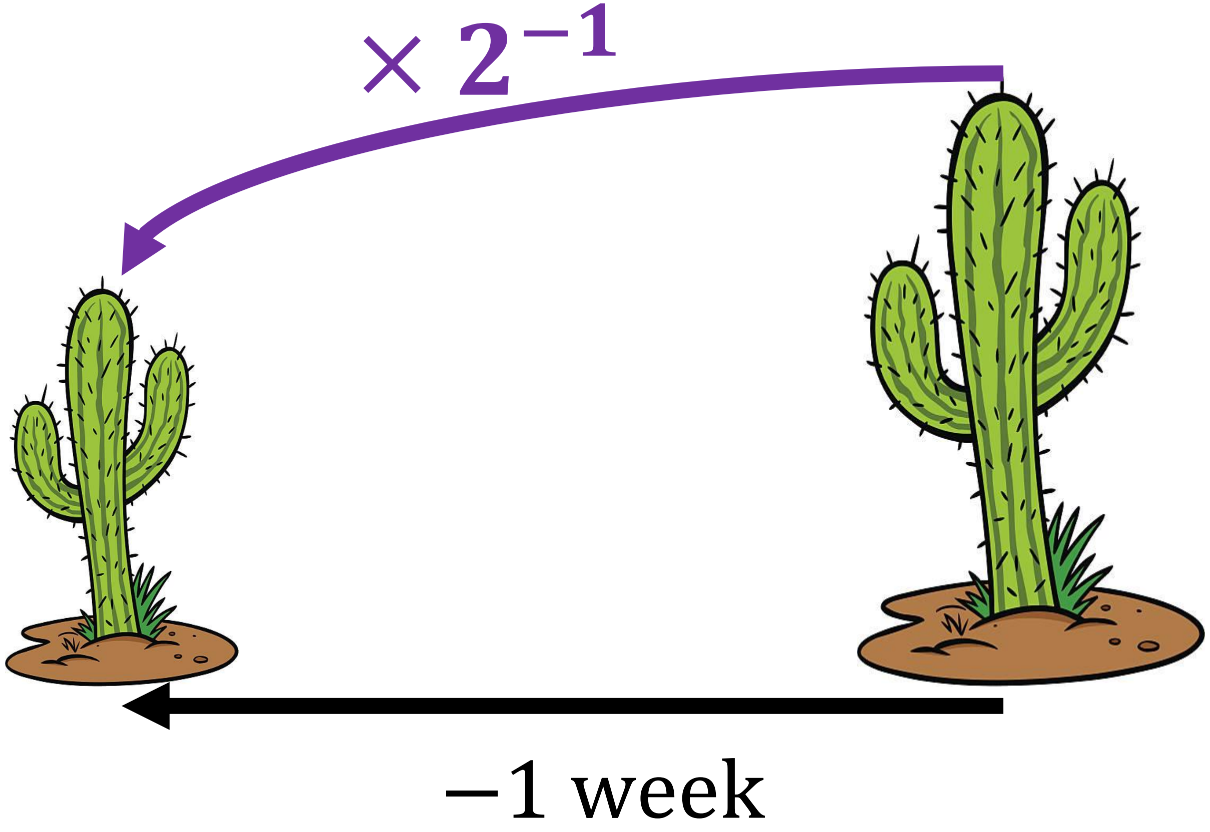
$$\mathbf{2^0 = 1}$$



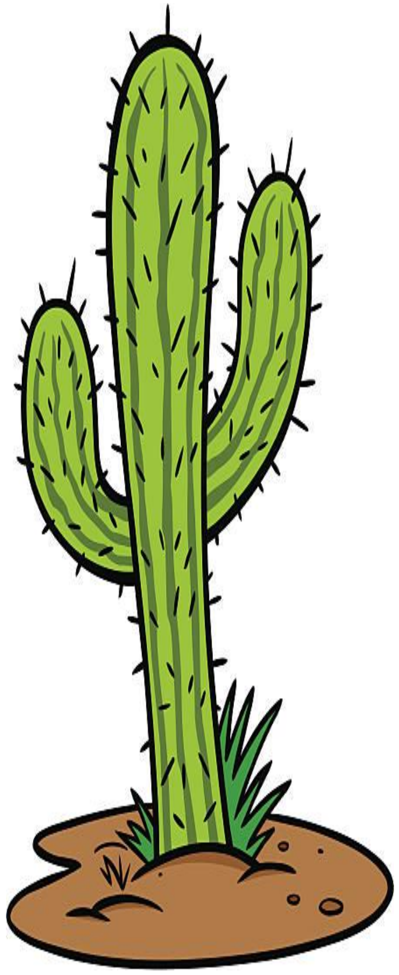
no time passes

$$\mathbf{2^0 = 1}$$

What does $\mathbf{2^{-1}}$ mean in this story?



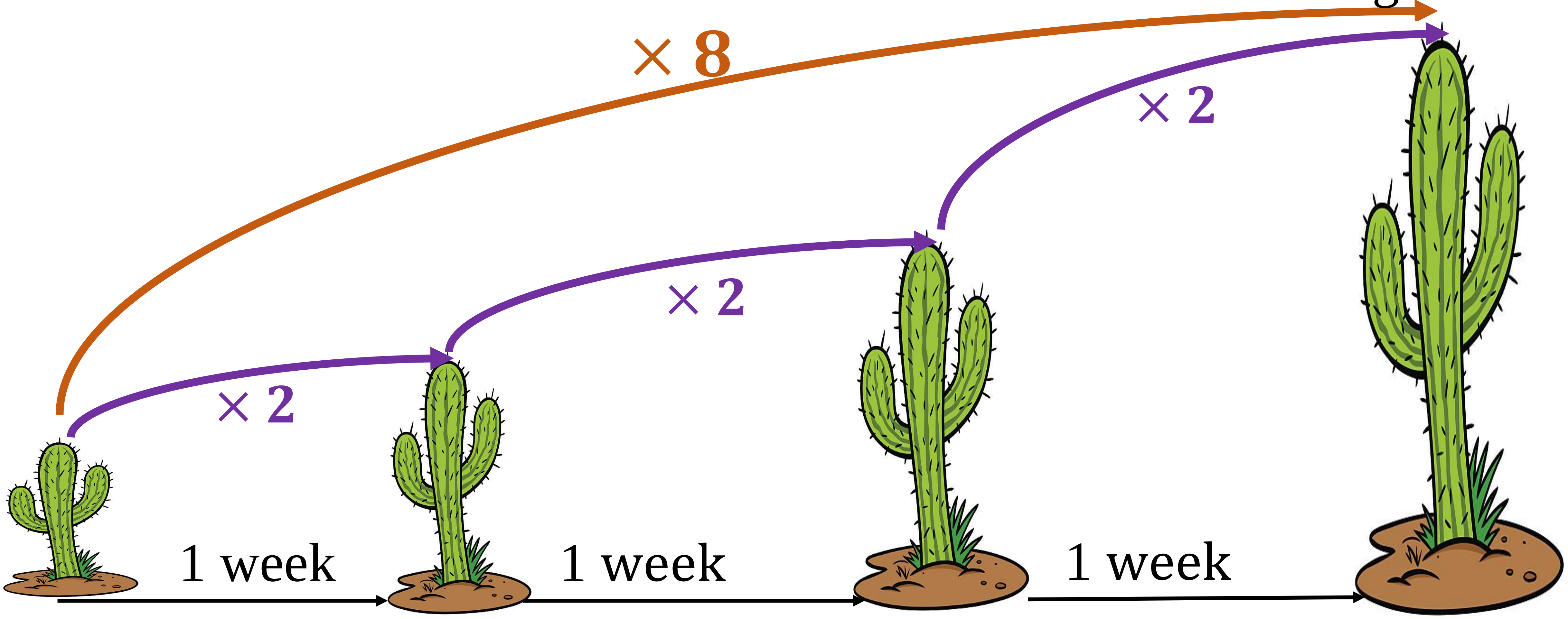
log



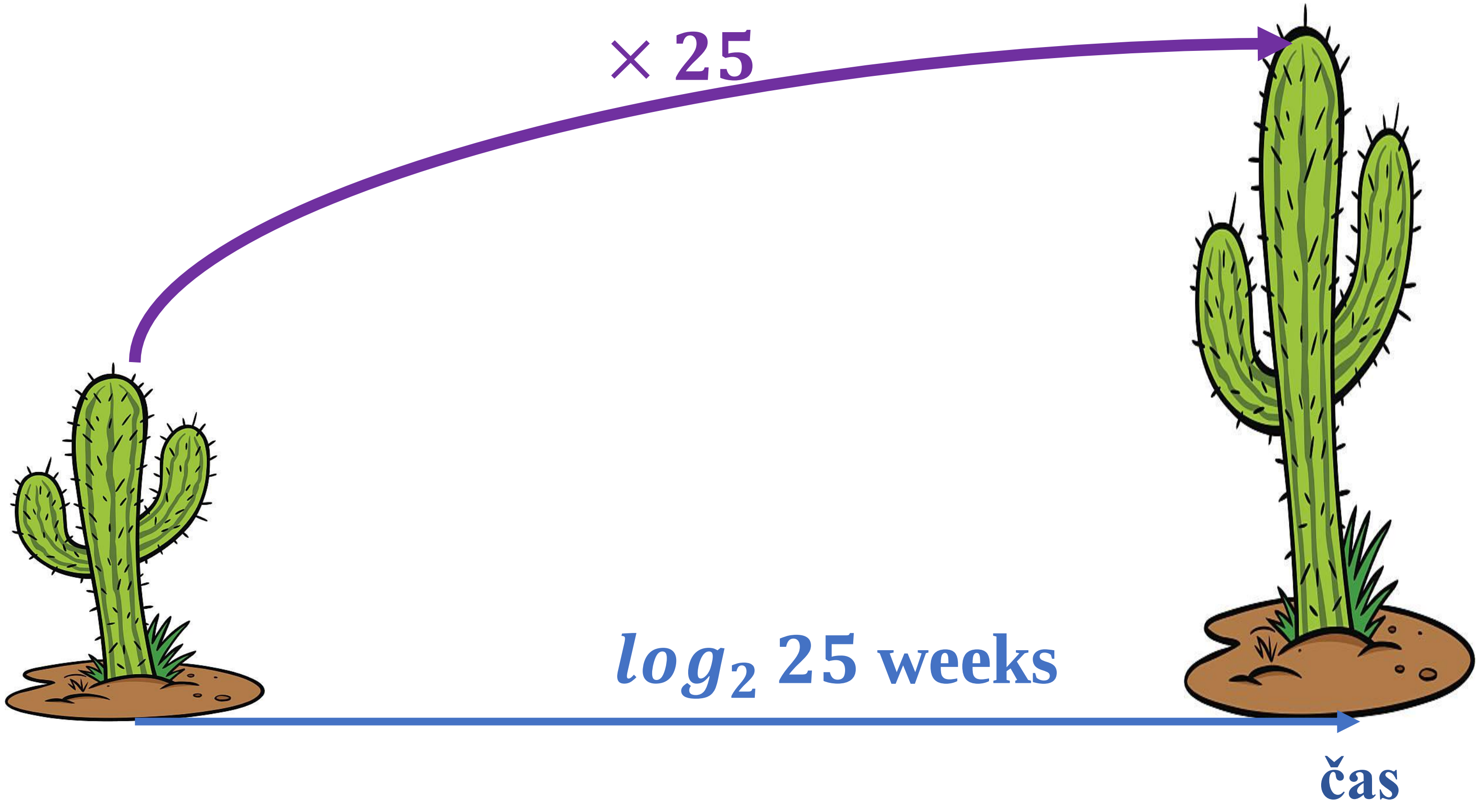
How can we describe $\log_2 8$ in the cactus story?

$\log_2 8$ is the number of factors of **2** that we have in the number **8**.

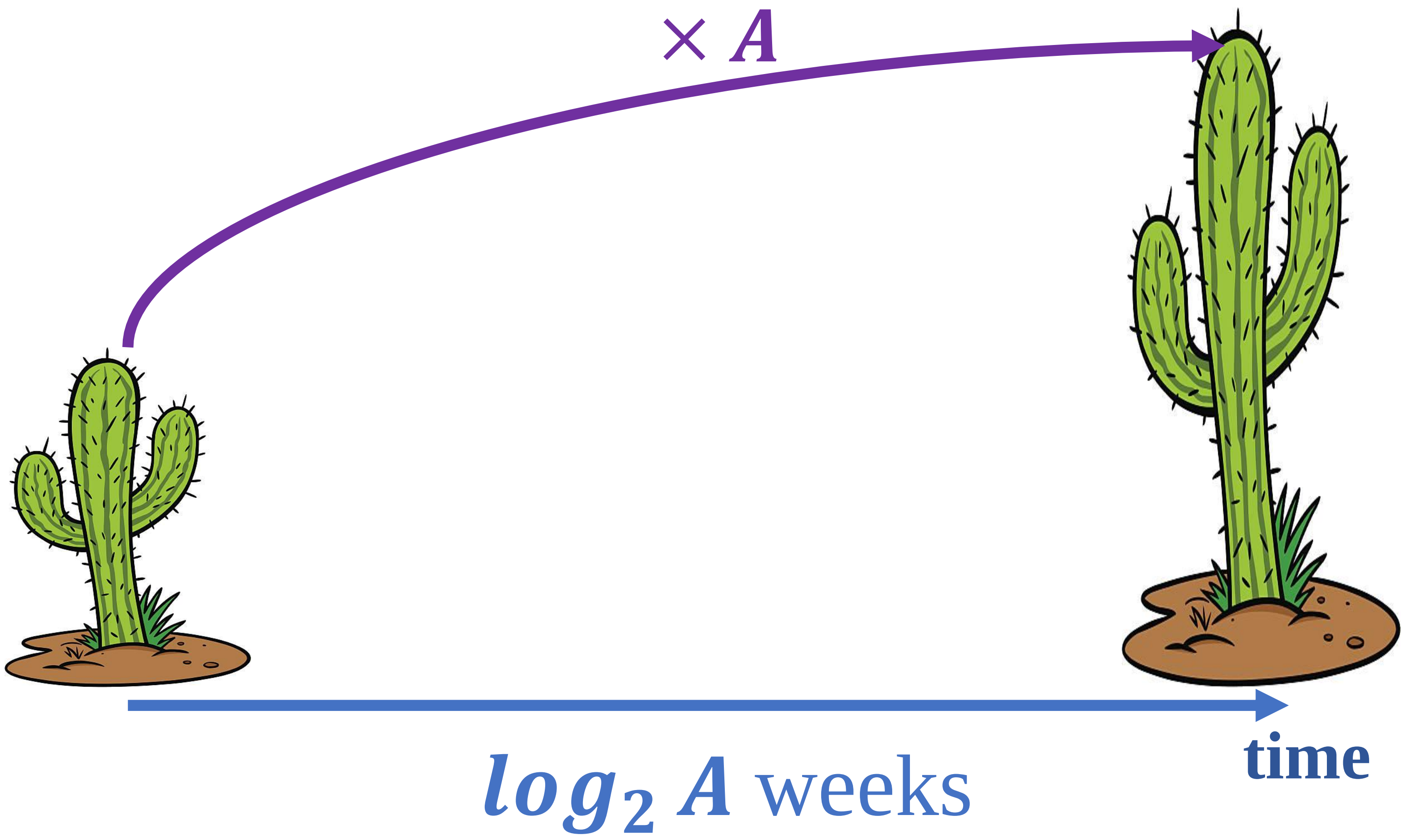
$\log_2 8$ is the number of weeks the cactus needs to reach **8 times** its height.



$\mathbf{log_2\, 25}$ is **the number of weeks** the cactus needs to reach **25 times** its height.

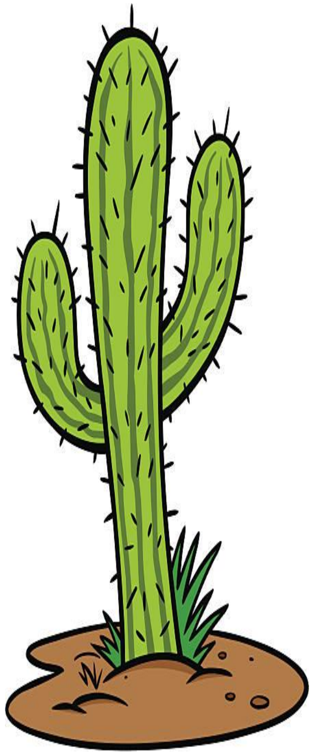


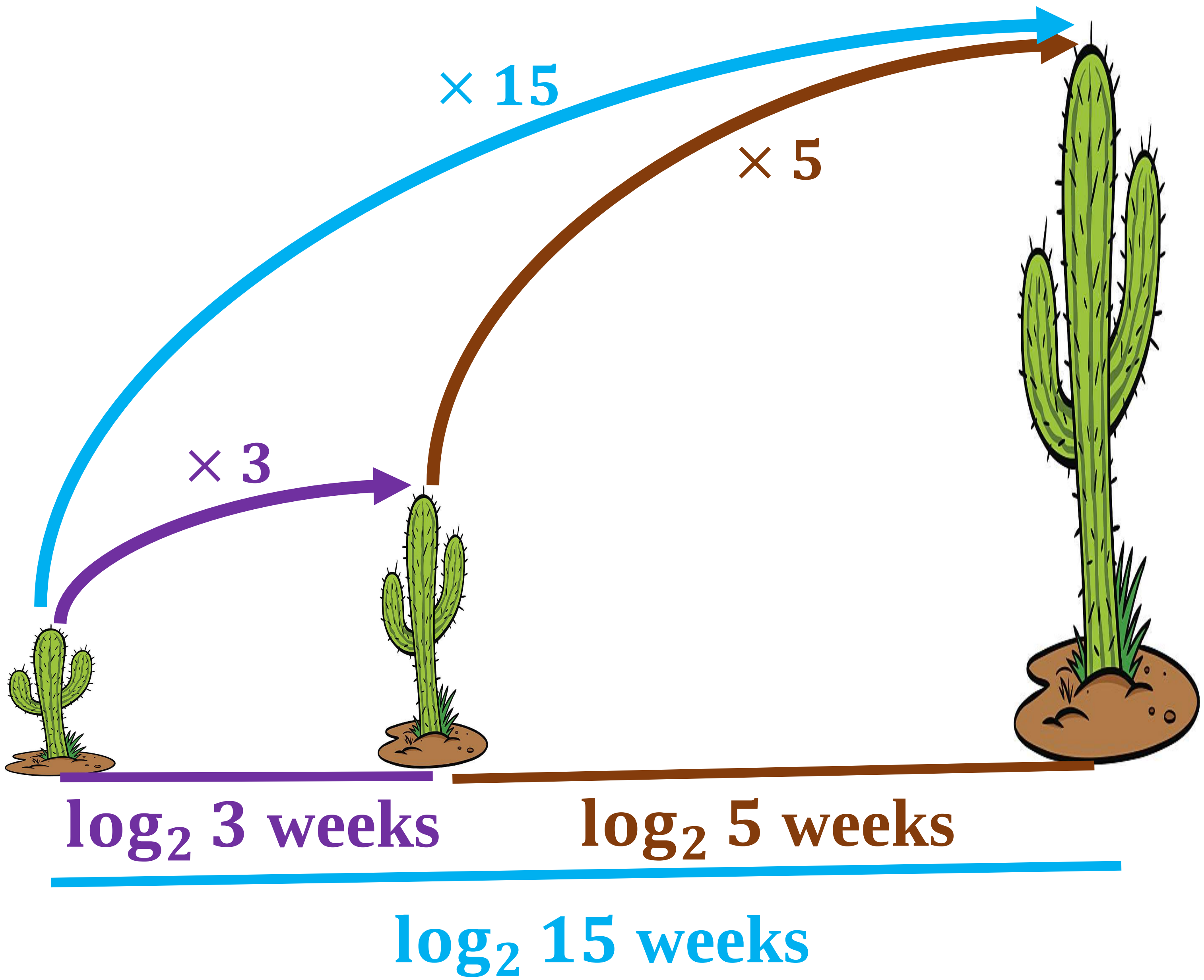
$\boldsymbol{log_2 A}$ expresses the number of weeks the cactus needs to reach **$\boldsymbol{A}$-times** its height.



$$\log_b X + \log_b Y = \log_b XY$$

$$\log_2 3 + \log_2 5 = \log_2 15$$

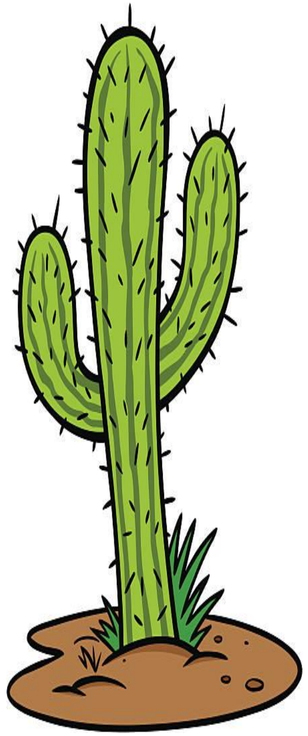




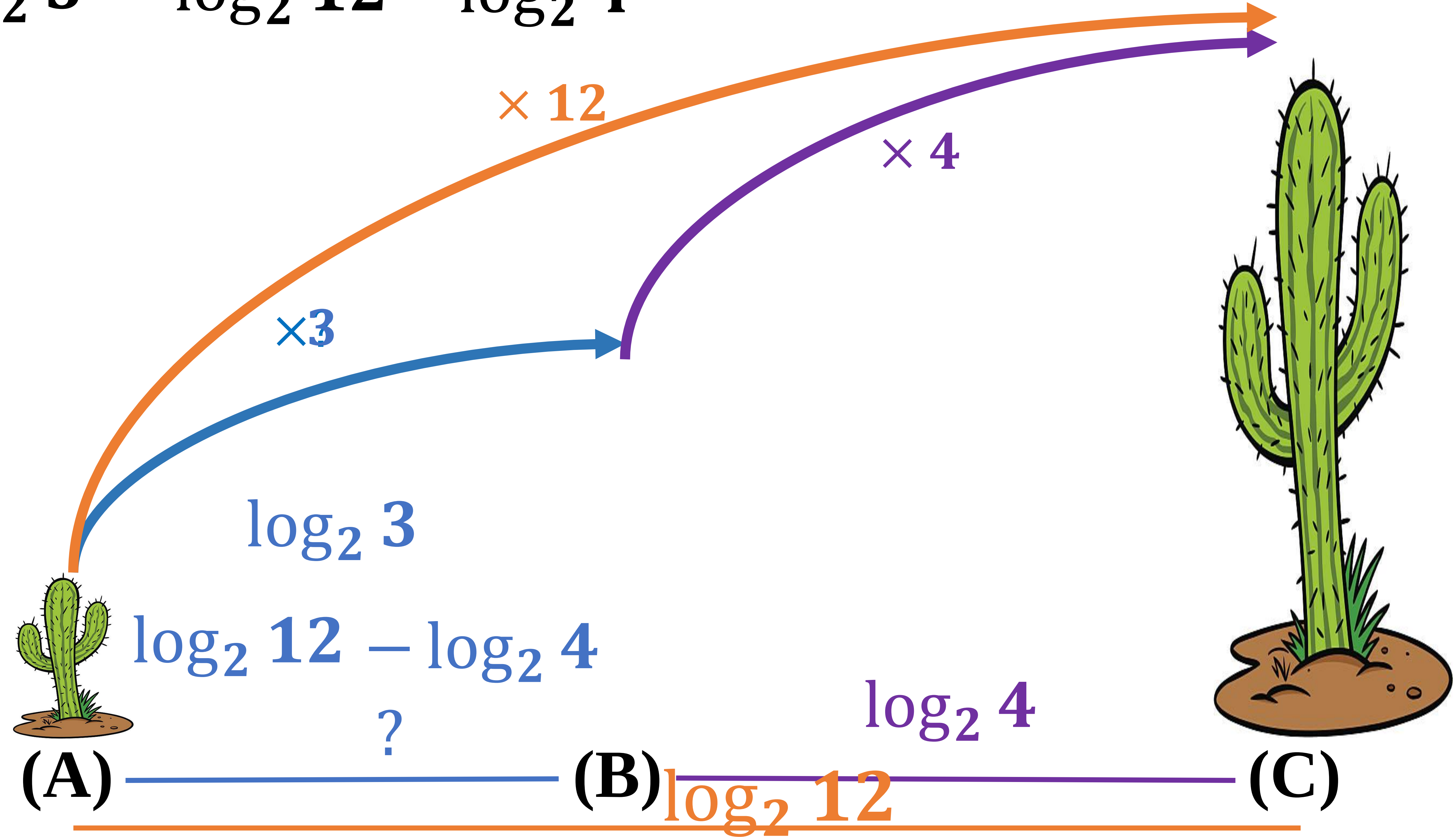
$$\log_2 3 + \log_2 5 = \log_2 15$$

$$\log_b X - \log_b Y = \log_b \frac{X}{Y}$$

$$\log_2 12 - \log_2 4 = \log_2 3$$

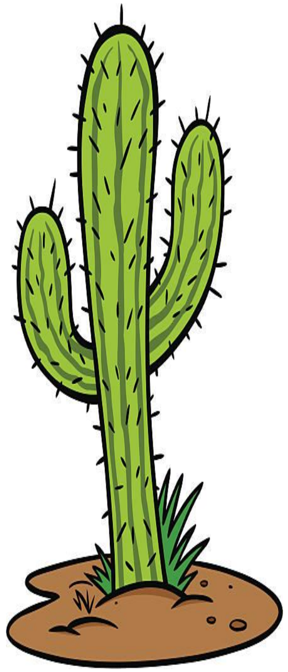


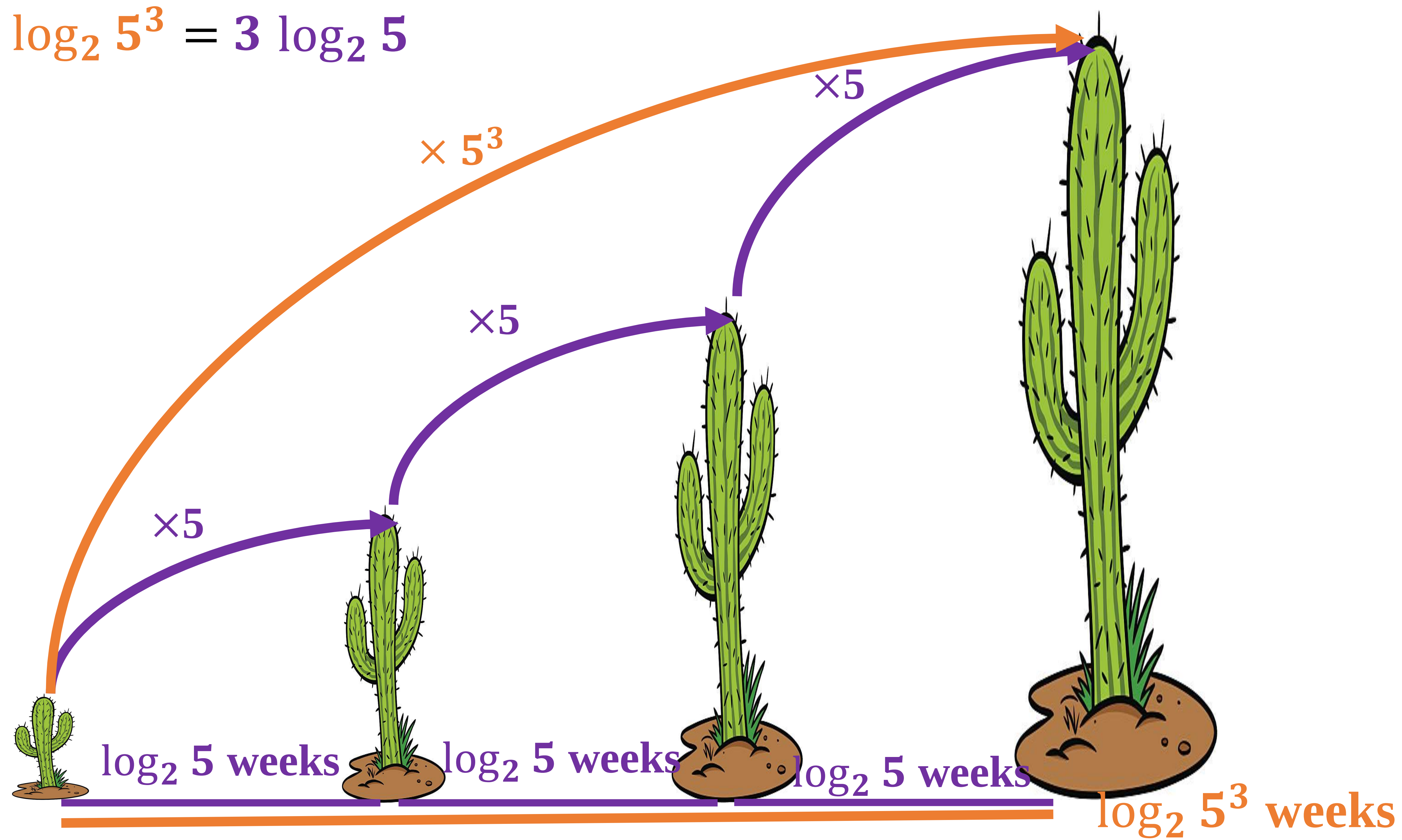
$$\log_2 \mathbf{3} = \log_2 \mathbf{12} - \log_2 \mathbf{4}$$



$$\log_b X^y = y.\log_b X$$

$$\log_2 \mathbf{5}^3 = \mathbf{3} \log_2 \mathbf{5}$$





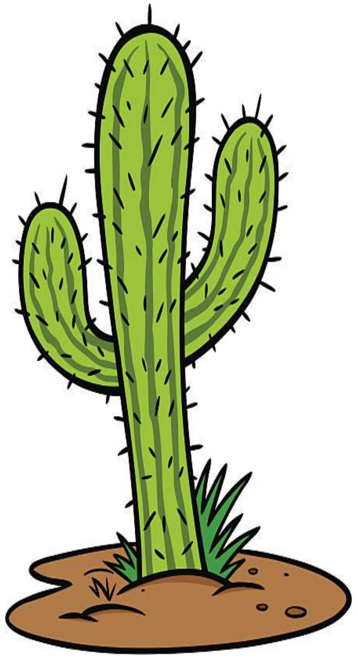
$\log_2 5^3 = 3\log_2 5$
$\times 5^3$
$\times 5$
$\times 5$
$\times 5$
$\log_2 5$ weeks
$\log_2 5$ weeks
$\log_2 5$ weeks
$\log_2 5^3$ weeks

Other logarithmic rules can also be interpreted using the cactus story.

$$\log_2 2^A = A$$

$$2^{\log_2 A} = A$$

$$\frac{\log_2 A}{\log_2 B} = \log_B A$$

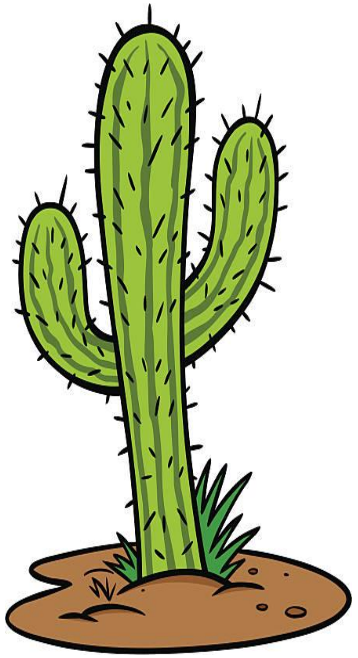


Další logaritmické věty můžeme interpretovat v příběhu o kaktusu.

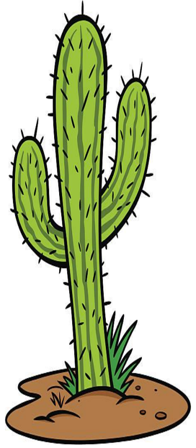
$$\log_2 2^A = A$$

$$2^{\log_2 A} = A$$

$$\frac{\log_2 A}{\log_2 B} = \log_B A$$



We have a cactus whose height doubles every week. Fill in the blanks.



a) The number of weeks the cactus needs to reach 2^3-times its height is …3

b) The number of weeks the cactus needs to reach 2^4-times its height is …4

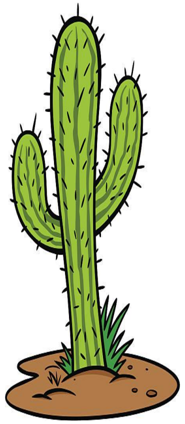
c) The number of weeks the cactus needs to reach 2^{100}-times its height is …100

d) The number of weeks the cactus needs to reach 2^A-times its height is …A

e) Write the statement from part (d) as a logarithmic equation.

$$\log_2 \boldsymbol{2^A = A}$$

We have a cactus whose height doubles every week. Fill in the blanks.



a) $2^{\text{the number of weeks the cactus needs to reach } 8-\text{times its height}}$ is 8

b) $2^{\text{the number of weeks the cactus needs to reach } 9-\text{times its height}}$ is 9

c) $2^{\text{the number of weeks the cactus needs to reach } 10-\text{times its height}}$ is 10

d) $2^{\text{the number of weeks the cactus needs to reach } 100-\text{times its height}}$ is 100

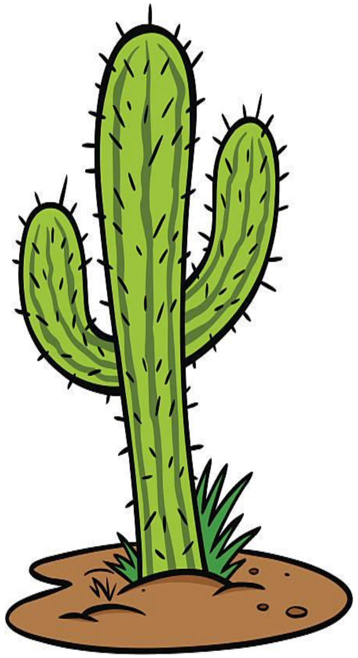
e) $2^{\text{the number of weeks the cactus needs to reach } A-\text{times its height}}$ is A

f) Write the statement from part (e) as a logarithmic equation.

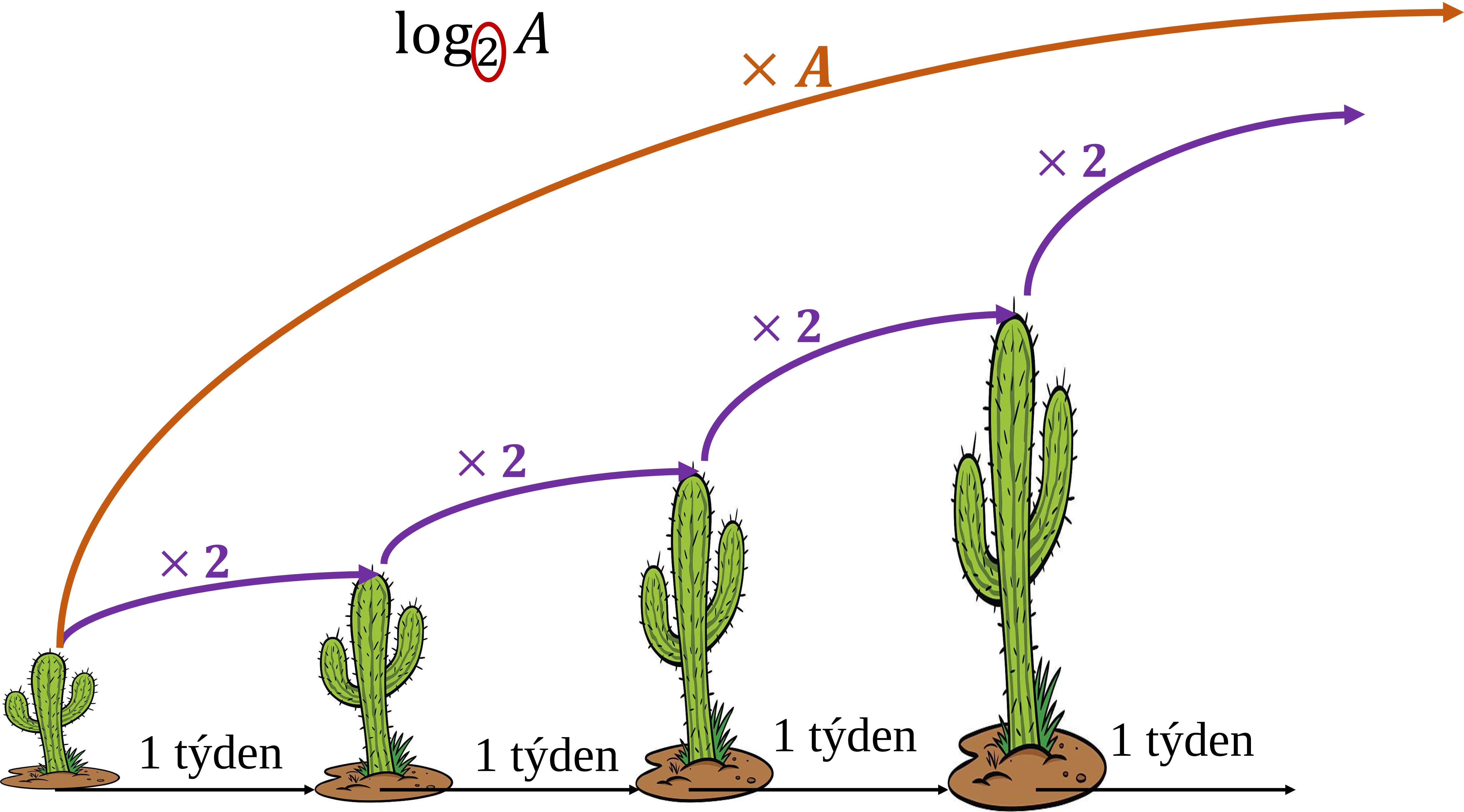
$$\mathbf{2^{\log_2 A} = A}$$

Use the cactus story and explain why this logarithmic statement is true.

$$\frac{\log_2 \boldsymbol{A}}{\log_2 \boldsymbol{B}} = \log_{\boldsymbol{B}} \boldsymbol{A}$$



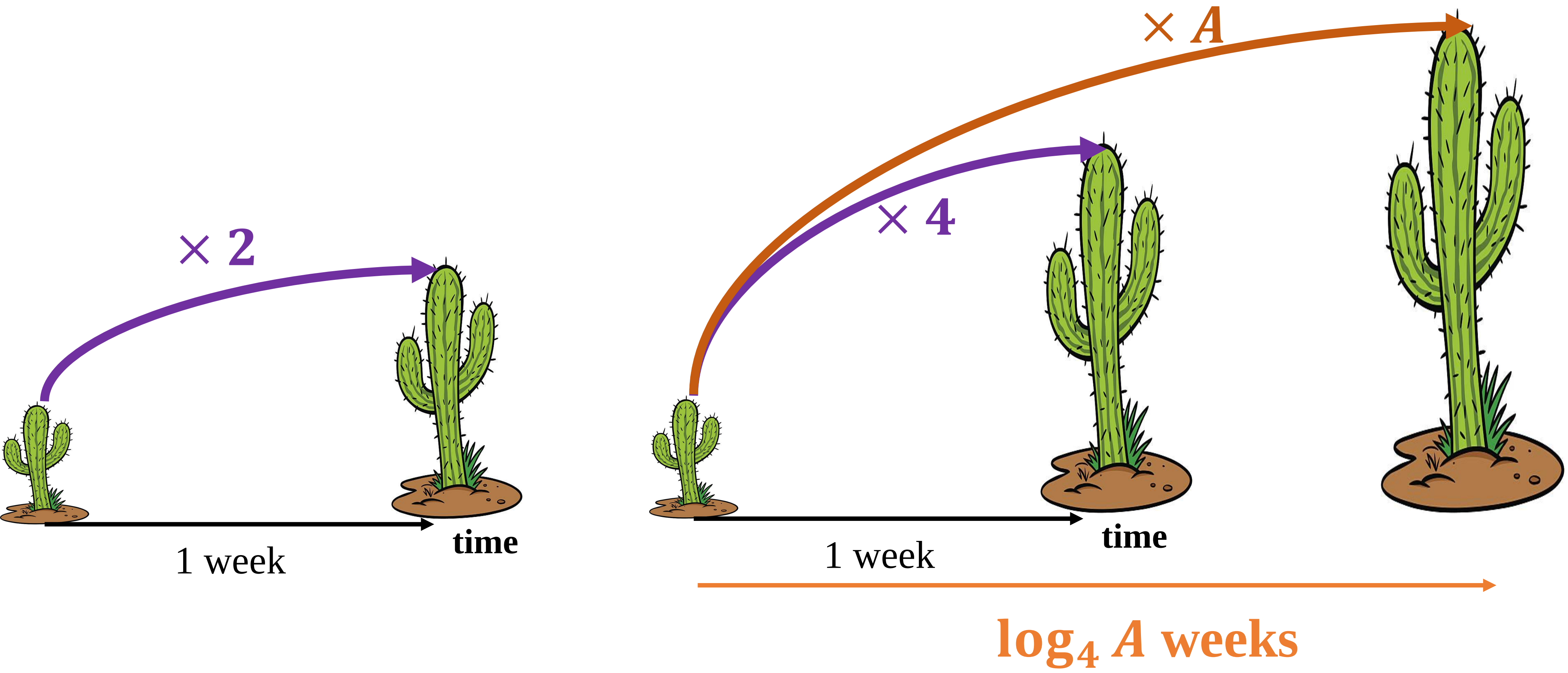
$$\frac{\log_2 \mathbf{15}}{\log_2 \mathbf{5}} = \frac{\log_4 \mathbf{15}}{\log_4 \mathbf{5}} = \log_5 \mathbf{15}$$

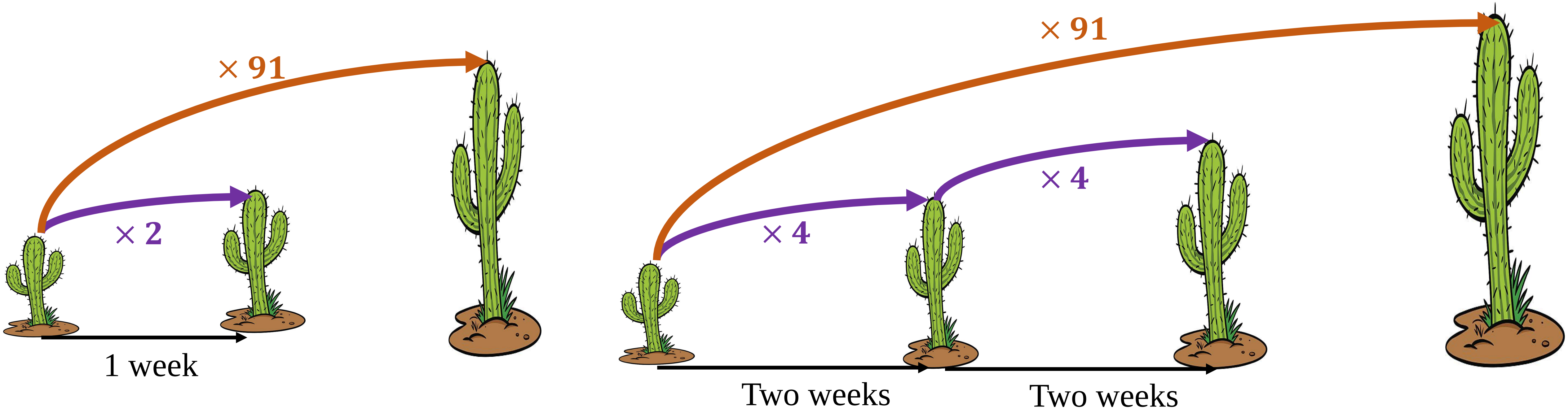


$\log_2 A$
× A
× 2
× 2
× 2
× 2
1 týden
1 týden
1 týden
1 týden

How can we change the base of the logarithm from 2 to another number (e.g., 4, 8, 16, etc.)?

$\log_2 A$ $\log_4 A$

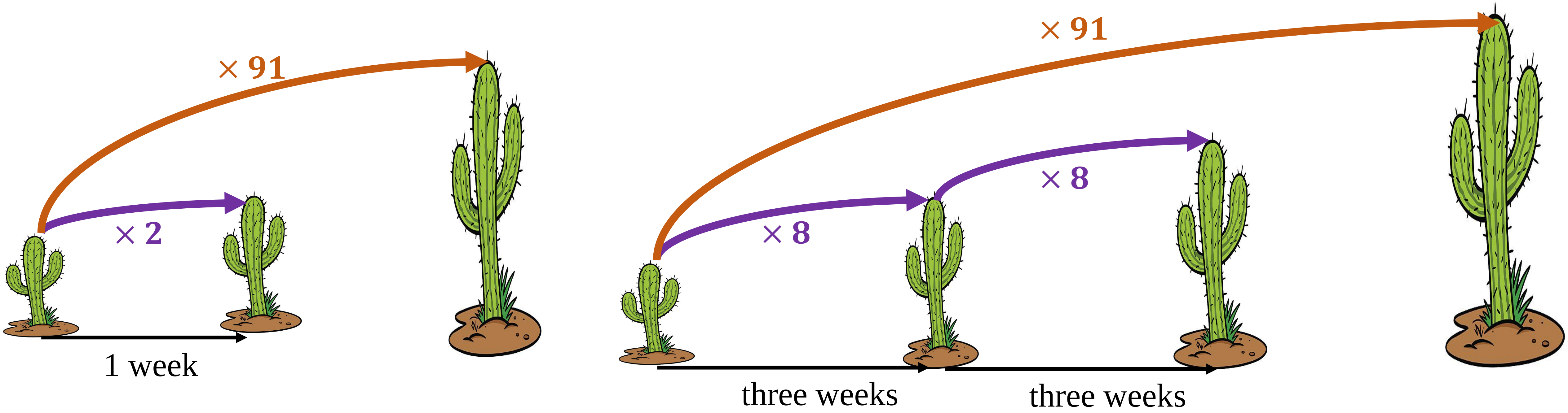




$\log_2 91$ represents the number of weeks the cactus needs to reach 91 times its height.

two-week periods

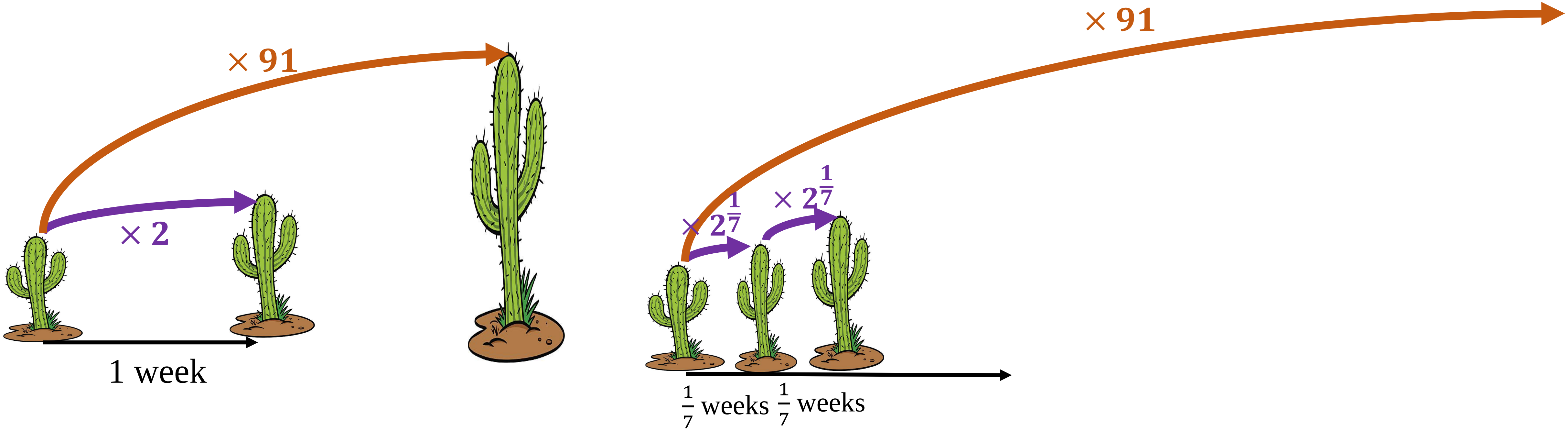
$\log_4 91$ represents the number of ………. the cactus needs to reach 91 times its original height.



$\log_2 91$ represents the number of weeks the cactus needs to reach 91 times its height.

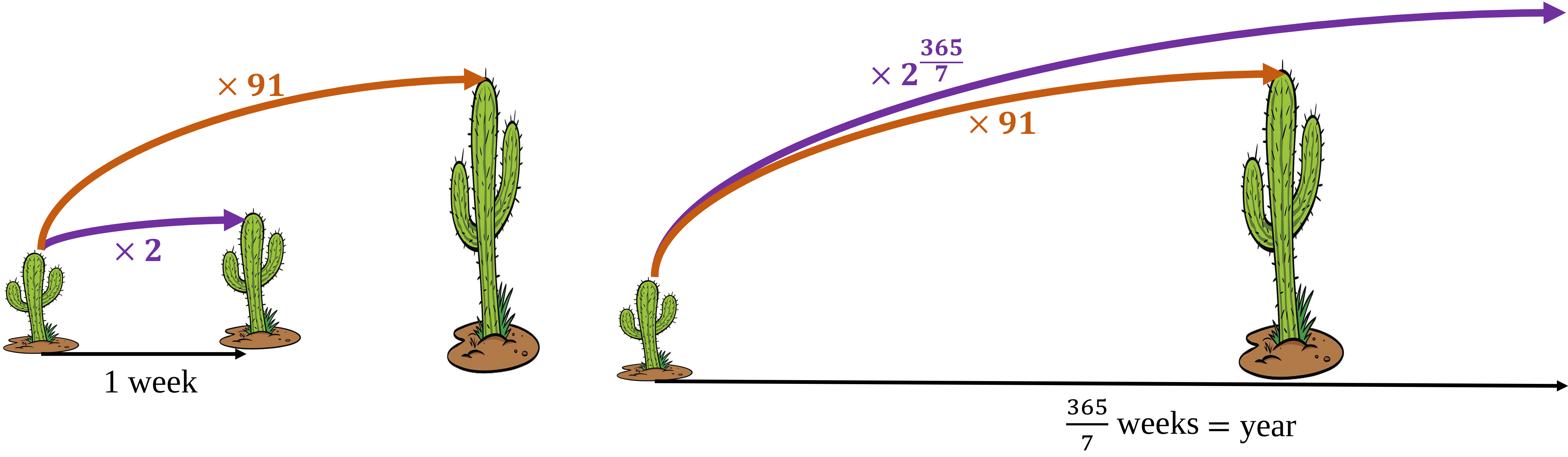
three-week periods

$\log_8 91$ represents the number of ………. the cactus needs to reach 91 times its height.



$\log_2 91$ represents the number of weeks the cactus needs to reach 91 times its height.

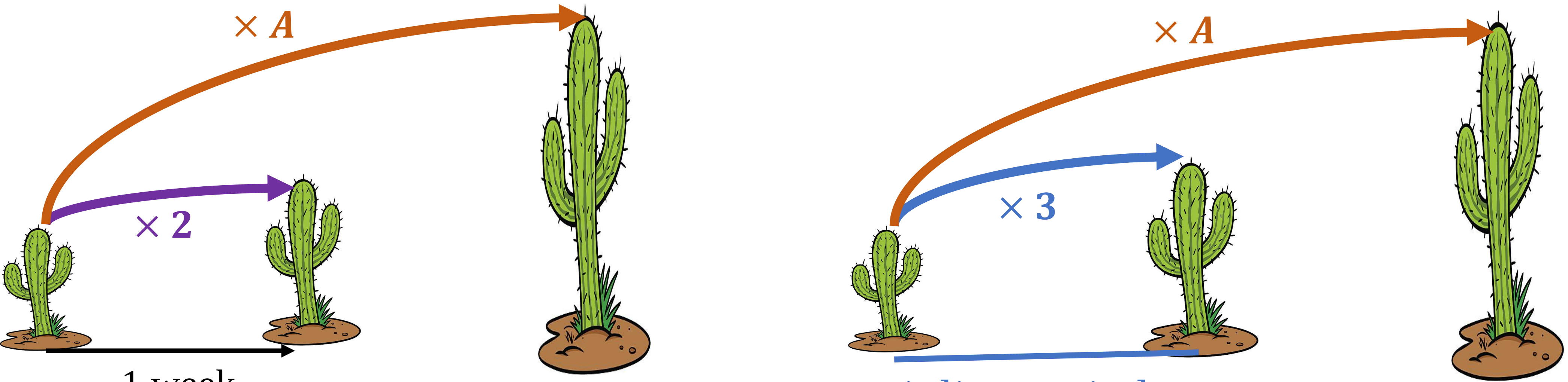
$\log_{2^{\frac{1}{7}}} 91$ represents the number of ………. (days) the cactus needs to reach 91 times its height.



$\log_2 91$ represents the number of weeks the cactus needs to reach 91 times its height.

$\log_{2^{\frac{365}{7}}} 91$

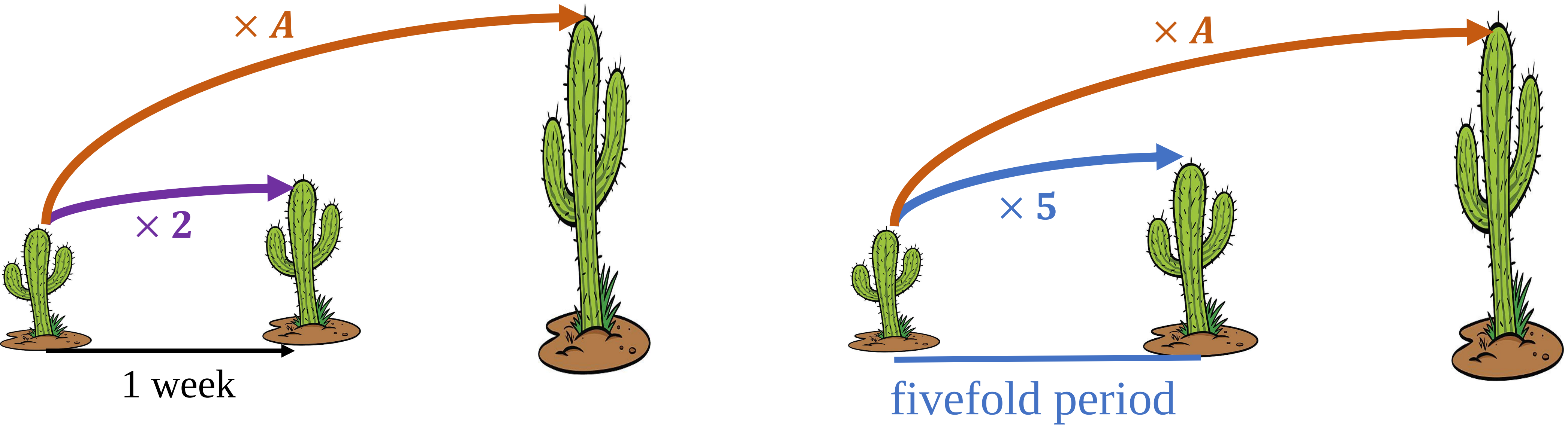
represents the number of ……….. **years** the cactus needs to reach 91 times its height.



tripling period

= is the amount of time it takes for the height of a cactus to triple.

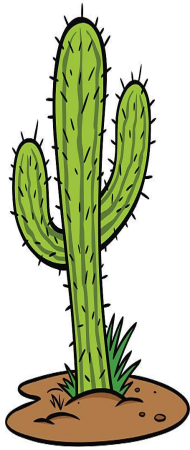
$\log_3 A = ?$ How many **tripling periods** does the cactus need to reach A *times* its height?



= is the amount of time it takes for the height of the cactus to increase fivefold.

$\log_5 A = ?$ How many **fivefold periods** does the cactus need to reach A *times* its original height?

We have a cactus whose height doubles every week. Fill in the blanks.

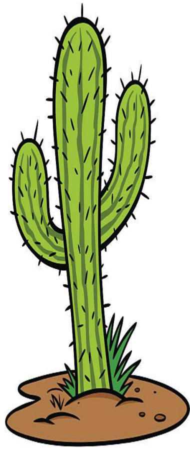


a) A doubling period is the time …..it takes for the cactus height to double.

b) A quadrupling period is the time …it takes for the cactus height to become four times as large.

c) A tenfold period is the time …it takes for the cactus height to become ten times as large.

We have a cactus whose height doubles every week. Answer the following questions (your answer should be a natural number). For each part, draw a diagram and show how you can use it to justify your answer.



a) How many doubling periods does the cactus need to reach *eight times* its original height? 3

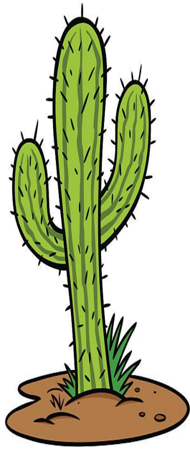
b) How many quadrupling periods does the cactus need to reach *sixteen times* its original height? 4

c) How many sixfold periods does the cactus need to reach *216 times* its original height? 3

d) How many tripling periods does the cactus need to reach *81 times* its original height? 4

e) How many fivefold periods does the cactus need to reach *five times* its original height? 1

We have a cactus whose height doubles every week. Answer the following questions (your answer should be expressed in logarithmic form).



How many doubling periods does the cactus need to reach nine times its original height? $\log_2 9$

How many quadrupling periods does the cactus need to reach five times its original height? $\log_4 5$

How many sixfold periods does the cactus need to reach 200 times its original height? $\log_6 200$

How many tripling periods does the cactus need to reach eight times its original height? $\log_3 8$

$$\boxed{\log_{y^a} x = \frac{1}{a}\log_y x}$$

Homework: We have a cactus whose height doubles every month. Using this cactus, design a simple word problem that demonstrates that the equality $\log_{2^{12}} 17 = \frac{1}{12}\log_2 17$ holds.

Hint: You must answer your word problem in two different ways. In the first method, you must obtain $\log_{2^{12}} 17$ as the answer. In the second method, you must obtain $\frac{1}{12}\log_2 17$. Since the word problem has only one correct answer, you may conclude that the equality $\log_{2^{12}} 17 = \frac{1}{12}\log_2 17$ holds.

References

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Děkuji za pozornost!

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